MATH V23 LECTURE NOTES (Bowen)  
Review of Multivariable Calculus Concepts

One way to write the equations we will be solving in sections to come is in the form . Because the expression on the right side of this equation is a function of two independent variables, we will need to introduce (or review, if you’ve already taken MATH V21C) an extension of the basic ideas of calculus, as they apply to functions of two or more variables.

First derivatives of a function of several variables. A differentiable function has as many distinct first derivatives as it has independent variables (that is, one derivative per independent variable). These derivatives are called ***partial derivatives*** because each derivative describes only that portion of the total rate of change of the function that is caused by changing the value of any *one* of the independent variables, while holding the values of all the other independent variables fixed (constant). For a function of two variables , the derivatives with respect to *x* and *y* are written as  and , or sometimes as  and, respectively. (The prime notation [*e.g.*, ] should *not* be used to represent partial derivatives, as it would not be clear whether  represented  or . If a compact notation analogous to is desired, use  and instead.)

The derivative  describes the rate of change of *f* when only *x* is changed while *y* is held constant. The derivative  describes the rate of change of *f* when only *y* is changed while *x* is held constant. Computing partial derivatives, in the light of these definitions, is straightforward; all the ordinary rules of derivatives (power rule, product and quotient rules, chain rule, *etc*.) still apply, but only for the “variable” whose value is changing. To compute, , we treat *x* as a variable, and all the other independent “variables” (just *y* in this case) as *constants*, that is, as if they were a 3 or a 7 or a −5; likewise, to compute , we treat *y* as a variable, and *x* as a constant. By way of example, let’s find the partial derivatives of . Starting with , we note that the first term () is a product of functions of the independent variable *x*, so we’ll have to use product rule. The second term () looks similar, but because it only contains *y*, the entire expression is effectively a constant that is added (not multiplied) with its neighboring terms, so its derivative will become zero. The third term () looks like another product rule at first, but with *y* being constant, product rule is unnecessary; the term is treated as if it were like , which means the *y* just sits out in front as a constant would, so we only differentiate . (The *y* does not go to zero because it is multiplied, not added, with .) The last term () might look like yet another product rule, but *y* is again a constant, so product rule is not needed here, either. The result is



As an exercise, see if you can show (by treating *y* as a variable and *x* as a constant) that



Note that if *x* is a constant, then so are larger expressions containing only *x*, like , , , , and so on.

Second derivatives of a function of several variables. By taking another partial derivative of one of the first partial derivatives found above, we may develop several types of second partial derivatives. Continuing with the function  introduced above, we find that





It is also possible to construct a different type of second derivative by taking the derivative with respect to *y* of the derivative with respect to *x*, or vice versa. These second derivatives are ***mixed partials***, and it has been proved that they have an interesting property: if the mixed partials are continuous at a given point, then they are equal there, regardless of the order in which they are taken. We may illustrate this by computing the mixed partials of :





This property is essential to the development of the ODE-solving technique we will develop later in this section. Note how the notations  and  both indicate that the derivative with respect to *x* is taken first, and the derivative with respect to *y* is taken second, even though the order in which *x* and *y* appear in these notations is different.

Integration of a function of several variables. As with partial differentiation, integration may be conducted with respect to any one of the independent variables; the other variables are treated as constants. For a function *g* of two independent variables, integration/antidifferentiation with respect to *x* is written as , and integration with respect to *y* is written as .

As with single-variable integration, multi-variable integration, when applied to the derivative of a function, should restore the original function, in accordance with the Fundamental Theorem of Calculus. In other words, we should find that , and . Let’s try integrating the derivatives of the function  whose partial derivatives we’ve already computed previously, to test this idea.



To evaluate the three separate integrals, we note that  is the derivative of , and that the factors of *y* in the last two integrals are constants (since we are integrating with respect to *x*), which may therefore be factored out of their respective integrals. This leads to



The latter expression should be equal to , and it almost is, except that the term  has gone missing. Could we have made an error? Let’s try integrating  with respect to *y* to see whether that result can help us track down the error:



We note that  is the derivative of , and that the quantities x and  are constants (since we are integrating with respect to *y*). The solution is



The latter expression should also be equal to , and again, it almost is, except this time the term  has gone missing. What is unique about the missing terms? The integral with respect to *x* is missing a term that only contains *y*, and the integral with respect to *y* is missing a term that only contains *x*. In both cases, we find that the missing terms are *constants* in the context of their respective integrals. So, our error was an oversight; instead of writing the constants of integration as *C*, we should have written them as functions of the variable(s) that were treated as constant(s) during the integration process. These constants still disappear (as a *C* would) when the corresponding partial derivative is computed to test the results of each integration. The correct way to rewrite the above results would therefore be





Then, by setting  and , we find that each of the above integrals really does restore the original function  plus a constant *C*. In general,  and  may be determined by writing the results of both integrals near each other (as above), and noting which terms appear in one result but are missing in the other. In the above expressions,  appears in the first result, but is missing in the second, so it becomes  (after appending  to it); and  appears in the second result, but is missing in the first, so it becomes  (after also appending  to it).

Differentials of a function of several variables. The slope (derivative) of a line or a curve is given locally by



(Technically, a curve can’t have a slope; the “slope” of a curve is really the slope of a tangent line to the curve, but that distinction is not important for this discussion.) If we turn the above relationship around using algebra, we may compute the “rise,” assuming we already know the slope and the “run,” by multiplying the above equations on both sides by “run” to give



Using calculus notation to rewrite the last result above gives (for a function of one variable)



The “rise” thus calculated (*dy*) is called a ***differential***. (The run *dx* is also a differential, but we are usually more interested in the rise.) This is distinct from a derivative, because the derivative () represents the ratio of both rise and run, whereas the differential only gives the value of the rise, or of the run. The above formula for the differential should appear somewhere in your MATH V21A notes, and you’ve also used it in integration as part of the *u*-substitution process.

We must modify this calculation slightly for functions of more than one variable. One way to model a function of two variables  is with a topographical (terrain elevation) map, such as the “lumpy” tactile ones, typically stamped into hard plastic, that you might have encountered in a geography or geology course. On this surface, let the *x*-axis be a line running in the east-west direction along the surface of the map, the *y*-axis be a line running in the north-south direction, and *z* a measure of the elevation above sea level of any given point on the map (which obviously depends on the values of *x* and *y*). On hilly terrain, we may change our elevation *z* (the rise) by walking a short distance *dx* (the run) in the *x*-direction, or a short distance *dy* (also the run, but along a different axis) in the *y*-direction, or both. If we only walk in the *x*-direction, then *y* is constant, which means we may adapt the one-variable definition of the differential by substituting the partial derivative  to represent the slope. This leads to



as the formula to compute the change in elevation *dz* resulting from walking only in the *x*-direction. If we only walk in the *y*-direction, we obtain



If we walk in both the *x*- and *y*-directions, there are two contributions to the elevation change: one from *dx* and one from *dy*. For very small changes, these two effects are additive, and we find that the total differential (the change in elevation obtained from the combination of *dx* and *dy*) is



This is the source of equation (1) in the textbook [Zill]. In this equation, *dz* is called the ***full differential*** or ***exact differential*** of *z*. Because *f* and *z* are interchangeable, this may also be written



Supplemental homework exercises. Please complete the following problems before you start working on the main homework assignment for section 2.4.

1. For the function , compute the following derivatives:
2. 
3. 
4. 
5. 
6. For the function  in the preceding problem, compute the mixed partials  and , and show they are equal. Hint: use the results of problems **1(a)** and **1(b)** above to expedite your work.
7. Integrate the results of  and  from problems **1(a)** and **1(b)** above, and show that they can be combined to yield a full antiderivative in the form .
8. Find an expression for the full differential *df* of the function given in problem **1**.