MATH V23 LECTURE NOTES (Bowen)  
Section 8.2  
Homogeneous Linear Systems

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Terms and concepts. Homogeneous system, square matrix, column vector, scalar, eigenvalue, eigenvector, trivial solution, characteristic equation.

Introduction. We seek to use matrix methods to help us solve linear first-order homogeneous systems of ODEs. A system is ***homogeneous*** if none of the equations in the system contains a term that depends explicitly on the independent variable, which is taken as *t*. (That is, a system is homogeneous if all the occurrences of  in equation (3) of section 8.1 of the textbook [Zill] are equal to zero.) In the context of this section, we assume that the independent variable is *t*, and the dependent variables are either *x*, *y*, *z* (typically if there are only two or three of them), or , , , ,… (typically if there are four or more of them).

Some terminology: If a matrix has the same number of rows and columns, it is called a ***square matrix***. If a matrix has exactly one column, it is sometimes called a ***column vector*** (a matrix with one row is sometimes called a row vector). In discussions (such as this) that use column vectors but no row vectors, the phrase ***column vector*** is frequently abbreviated to just ***vector***. A ***scalar*** is a number (such as 5 or ) or a mathematical expression (such as ) that is not an element of a matrix or vector.

Some theory: Following in the footsteps of the theory developed in chapter 2 of the textbook for first-order equations with constant coefficients, we assume that the solutions of the system will be linear combinations of functions of the form , where in general  is a complex constant. Due to time constraints, however, we will only explore the portions of this section that deal with real values of . As an example, the *solution* for a system of two linear first-order homogeneous equations involving the dependent variables *x* and *y* might look like

 (1)

Because the solutions for *x* and *y* contain the same constants , , and the same exponential functions and , it is convenient to write these solutions in matrix form, using , as

 (2)

where the top row of each matrix represents the coefficients of the solution for *x*, and the bottom row of each matrix represents the coefficients of the solution for *y*. A complete solution requires us to obtain both the constants  and  in the exponential functions (in the language of matrices, these are called ***eigenvalues***), and the matrices  and  (which are called ***eigenvectors***). The prefix *eigen-* is a German word that means “one’s own”; the German phrase *mein eigenes Buch* means “my own book.”

To generalize the above expression, it is possible to use  to represent  and  to represent , which leads to the previous equation being written as

 (3)

The set  is independent, and therefore constitutes a fundamental set for the system its elements solve. As usual, the coefficients  and  represent arbitrary constants of integration.

Example: Homework problem #3. We did this problem correctly in lecture, except I overlooked a small detail near the very end, thereby fooling myself into thinking I’d made an error. Here is the proper way to solve this problem. The homogenous system is

 (4)

Because we have already defined , we take the derivative of both sides (where the prime notation is understood to indicate ) to write , which we use to represent the left side of the above system (that is, the expressions to the left of the equals signs). We use the definition of the matrix product to write the right side of the system as , where and . The system may therefore be written formally as

 (5)

or in detail as

 (6)

We seek members of the fundamental set, each of which should be similar in structure to the expressions  and  that were introduced in the paragraph preceding equation (3). Since we don’t yet know either the eigenvalue or the eigenvector, we introduce the eigenvector as  and the corresponding eigenvalue as , and write a generic solution as

 (7)

from which it follows (since  and , and therefore , are constants) that the derivative of the assumed solution is

 (8)

Plugging these last two equations into equation (5) gives

 (9)

Although matrix quantities cannot be divided, it is legal to divide both sides by the scalar . Moving the scalar  to the front of the left side yields

 (10)

We would like to factor out the matrix , but the subtraction of the matrix  and the scalar  is not defined. We introduce the identity matrix , which has the property , to convert the second term from a scalar product to a vector product:

 (11)

Now we may factor out the  matrix from both terms. (We must be careful to write  on the right side, since it is on the right in both terms above; the reason is that matrix multiplication is not commutative. If  and  are matrices, then in general . One exception is the identity matrix ; if  is a square matrix, then .) The result of the factoring is the homogeneous algebraic system

 (12)

which is just the classic formula used in linear algebra to find the eigenvalues  and eigenvectors  of matrix . The above equation, if written out in detail for this example, looks like

 (13)

One obvious solution to this last homogeneous matrix system would be , which is the so-called ***trivial solution***. However, we seek nontrivial (nonzero) solutions for the eigenvector . An important theorem from linear algebra states that nontrivial solutions only exist if the determinant of  is equal to zero. So, we compute the determinant of the large matrix in the last step of equation (13) and set it equal to zero:

 (14)

The last equation above is called the ***characteristic equation***; solving it yields the system’s eigenvalues:

 (15)

To find the corresponding eigenvectors (which would complete the solution), we substitute each of these  values back into the last step of equation (13), and solve for . Note that the eigenvectors are not unique; if  is any eigenvector, then so is any nonzero scalar multiple of the form , which means that the solution  of equation (13) will always contain at least one adjustable (arbitrary) matrix element (in this case, either  or ). The values of the remaining elements may then be written as expressions containing the arbitrary element(s); for example, if  were considered arbitrary, we might obtain something like .

Setting up the system of equation (13) by substituting in the first eigenvalue solution gives

 (16)

It’s usually easier to solve this last system by rewriting it as an augmented matrix (in which  is invisible, but its presence is implied), rather than as a matrix product. That version of the last system above looks like

 (17)

and we solve by applying a sequence of elementary row operations. In Gauss-Jordan elimination, the goal is normally to use the row operations to generate an identity matrix on the left. We will never get that far in an eigenvalue problem, however, due to the infinite number of possible solutions (eigenvectors) that are scalar multiples of each other, but we go as far as we can until we obtain one or more rows of zeros in the matrix (this will always happen). We start the row operations by multiplying the first row above by , and the second row by 2:

 (18)

Adding the first and second rows, and placing the sum in the second row, achieves the expected row of zeros at the bottom of the matrix:

 (19)

Translating the first row of the matrix back into an algebraic equation gives

 (20)

Treating  as an arbitrary constant, we solve for , showing that its value depends on the choice for . (We could also have assumed that  was constant; the results would have been the same either way.) We obtain

 (21)

As  is arbitrary, we simplify the arithmetic by making the choice , giving . An eigenvector corresponding to  is therefore

 (22)

We repeat this process with the second eigenvalue . Returning to the last system of equation (13), we substitute this eigenvalue to obtain

 (23)

The corresponding augmented matrix is

 (24)

and multiplying just the second row by  gives

 (25)

which is where I had to stop in lecture, as we had run out of time. What I should have done was to add the first and second rows, and placing the sum into the second row, giving

 (26)

and to convert the first row into an equation, giving

 (27)

The traditional method of setting  gives the result

 (28)

but, as  is an arbitrary constant, the fraction-averse student might wish to experiment with setting  to give the equally valid result

 (29)

We obtain the fundamental set by plugging each eigenvalue and eigenvector we obtained successively into equation (7). The first independent solution, from equations (15) and (22), is

 (30)

and the second independent solution, from equations (15) and (29), is

 (31)

Plugging these last two results into equation (3) gives the general solution, which is

 (32)

If necessary, this result could also be broken down into individual solutions for *x* and *y*:

 (33)

This result may be readily verified by substituting these expressions for *x* and *y* into the original problem (equation (4)).