MATH V23 LECTURE NOTES (Bowen)
Section 7.3
Operational Properties I

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Terms and concepts. Horizontal shift, first translation theorem, unit (Heaviside) step function, second translation theorem.

Introduction. The purpose of this section is to find theorems to help us calculate Laplace transforms  for certain classes of functions that are difficult or impossible to integrate using the definition from Definition 7.1.1 on page 279 of the textbook [Zill]. The motivation, besides saving ourselves the labor and tedium of integration, is to expand our tables of Laplace and inverse Laplace transforms (Theorems 7.1.1 and 7.2.1 in the textbook, respectively) to enable solving a greater variety of ODEs.

Aside on horizontal shifts (translations) of the graph of a function. You should recall from your precalculus course that if it is desired to shift the graph of a function  horizontally toward the right (that is, parallel to the *x*-axis) by a distance of *h* units, it is sufficient to replace each and every occurrence of  in the equation by , creating a new (shifted) function . (Leftward shifts are obtained by using a negative value of .) For example, if the function is , a parabola with vertex at , it is possible to move the vertex three units to the right (that is, to ) by creating a new function .

Because a Laplace transform  is a function of the independent variable *s*, the translation of a Laplace transform function  to the right, by a distance *a* along the *s*-axis (where *a* is a constant), would be written as , using the same reasoning as in the preceding paragraph (but with a different variable and constant). As the Laplace transform is defined for each value of *s* through the integral in Definition 7.1.1 in the textbook, the shifted function  would be obtained by replacing  with  in the defining integral as well. We therefore obtain  through

  (1)

which is the same as Definition 7.1.1 in the textbook, except we have generalized it by substituting  for . Note that the limits of integration, being values of *t* rather than values of *s*, do not change from what they were in the definition of .

Translation on the *s*-axis (First Translation Theorem). Suppose that  is one of the functions listed in the table of Laplace transforms in Theorem 7.1.1 in the textbook (or any other function whose Laplace transform  is already known). Also let *a* be any real constant. Then it is straightforward to show (using Definition 7.1.1 from the textbook) that the Laplace transform of the function  is

  (2)

where the step from the last expression on the first line to the right-hand expression on the second line is obtained via equation (1). In short, multiplying *any* function  by  causes its Laplace transform to shift by a distance *a* (rightward if , and leftward if ) along the *s*-axis. This result is the First Translation Theorem (Theorem 7.3.1 in the textbook).

Example: finding a Laplace transform using the First Translation Theorem. Let us find the Laplace transform  of  without performing any integration. We define  and  to obtain the form . We also note (using the linearity of the Laplace transform, and Theorem 7.1.1(a) and 7.1.1(b) from the textbook) that

  (3)

Then we find :

  (4)

Because it is easier to find the inverse Laplace transform of fractions having constant numerators (that is, numerators not containing the variable *s*), we will usually not use the LCD to combine this last result into a single fraction, but just leave it as it is above.

Inverse Laplace transforms using the First Translation Theorem. Writing the First Translation Theorem in reverse, we obtain

  (5)

provided that . To use this in practice, it is often necessary to manipulate a function of  until it looks like a function of  before taking the inverse transform. This is illustrated in the next example.

Example: finding the inverse Laplace transform of a rational expression containing repeated factors in the denominator. In the previous section of the textbook, we used partial fractions to find the inverse Laplace transform of a rational expression when there were no repeated factors in the denominator. That method would not be practical if repeated factors were present. However, the First Translation Theorem now makes such an undertaking feasible.

Let us find the inverse Laplace transform of . The denominator is not factorable, but any quadratic can be partially factored by completing the square, allowing us to write  (verify by multiplying out the denominator and comparing to the original expression for ). The  in the denominator suggests that this is a Laplace transform that has been shifted by  (since . We therefore attempt to rewrite each *s* in the numerator so it also looks like . Because the numerator is linear, we may write

  (6)

where *a* and *b* are undetermined real constants, as our first attempt to get the numerator to look as though it also contains . It requires only straightforward algebra (matching coefficients of *s* and constants on the left side of the above equation to their counterparts on the right side) to obtain specific values for *a* and *b*:

  (7)

This further refinement allows us to write  as

  (8)

We have achieved our intermediate goal, which was to rewrite  so that *every* occurrence of  has been manipulated to look like , but without changing the function, except in its appearance. We now ask the following question: If  represents some other function  that has been shifted to the left on the *s*-axis by  units to obtain , what was the *unshifted* version of ? The answer can always be obtained by changing  back to , from which we obtain

  (9)

Obtaining  is useful because (1) the inverse Laplace transform of  is relatively easy to find, and (2) since  is related to  through the horizontal shift

  (10)

we may use the First Translation Theorem to leverage this into finding the inverse Laplace transform of . We proceed with these two calculations, starting with the inverse Laplace transform of :

  (11)

Let , so ; then use Theorem 7.2.1(d) and Theorem 7.2.1(e) from the textbook to find the inverse transform :

  (12)

Now we leverage equation (10) and (setting ) the First Translation Theorem to find the inverse Laplace transform of :

  (13)

The unit (Heaviside) step function. Many mechanical and electronic processes can be modeled by the unit step function, also known as the Heaviside step function. The independent variable for this function is typically time *t*. This function has two possible values:  (off) and  (on). If the function transitions from “off” (0) to “on” (1) at time , then it is written as . This is a piecewise function, with

  (14)

Depending on the application, the domain of this function may be either  (in which case *a* is assumed positive) or  (in which case *a* may take on any real value). For practical reasons, only the first definition is used with Laplace transforms, so it is safe to assume that  in this section. See Figure 7.3.2 on page 298 of the textbook for a representative graph of this function; note that the exact behavior of the graph depends on the value of *a*. Some authors notate this function as .

Most commonly, the Heaviside step function is multiplied by another function . The product  is equal to just the value of , provided that ; however, the step function “zeros out”  toward the left side of its graph (for ), as illustrated in Figure 7.3.3 and Figure 7.3.4 on page 298 of the textbook. If it is also desired to “zero out”  on the right end of the graph as well (that is, for ), we write ; this can model turning a system on at time *a*, then turning it off again at a later time *b*.

Translation on the *t*-axis (Second Translation Theorem). Suppose we know the Laplace transform  of a function , and we would like to find the Laplace transform of . Because  is a piecewise function, we may rewrite the textbook’s Definition 7.1.1 as a piecewise integral, with the break between the pieces occurring at , as suggested by the definition of  from equation (14):

  (15)

For the first integral in the last line above, , and for the second integral, , so this simplifies to

  (16)

Apply the *u*-substitution , ,  (remembering to change the limits of integration) to obtain

  (17)

As both *a* and *s* are constants (technically, only *u* is a variable), the factor  may be moved outside the integral, leaving

  (18)

This result is the Second Translation Theorem. As a corollary, note that if  (for which, from Theorem 7.1.2, ), and we turn this function “on” at  instead of , the new Fourier transform is . But this is just the Heaviside step function, so

  (19)

Equation (18) is not always easy to use, as the function multiplied with the Heaviside step function is often not in the shifted form . To find the Laplace transform of a product in the form , it is possible to replicate the derivation leading to equation (18) with  replacing . At the *u*-substitution step,  becomes , and the alternative form of the Second Translation Theorem becomes

  (20)

For inverse Laplace transforms, equation (18) allows us to rewrite the Second Translation Theorem as

  (21)

provided that .

Example: Laplace transform involving a step function. Find the Laplace transform of . We note that the step function “turns on” at time , but the factor  has not been shifted to this same start time. We have two alternatives; Method 1 would be to use a trig identity to rewrite  as a shifted quantity to match the step function, then apply equation (18); and Method 2 would be to use the alternative formula of equation (20). Let’s try it both ways to compare results.

**Method 1**. Modify  using the identity  to obtain . Both functions are now shifted by , so equation (18) applies. Using this equation and Theorem 7.1.1(e) from the textbook (with  and ) gives

  (22)

**Method 2**. Use equation (20) directly, with  and :

  (23)

Both methods give the same result, as expected.

Example: Inverse Laplace transform involving a step function. Find the inverse Laplace transform of . From the left side of equation (21), we see that we need to convert this expression into the form . This is readily accomplished if we set  and . Unfortunately, there are no inverse transforms for  listed in Theorem 7.2.1 in the textbook, so we manipulate  into a more manageable form using partial fractions. We obtain

  (24)

Setting  gives ; setting  gives , so ; and setting  gives , so  and . The partial fractions expression above becomes

  (25)

The expression for the inverse transform becomes

  (26)

Define , , and ; then

  (27)

which, according to equation (21), simplifies to

  (28)

From Theorem 7.2.1 in the textbook, we find that

  (29)

Substituting these results into the expression for the inverse Laplace transform gives

  (30)

The textbook gives additional examples showing how these transforms and inverse transforms may be used to solve differential equations, including IVPs and boundary-value problems.