MATH V23 LECTURE NOTES (Bowen)
Section 7.2
Inverse Transforms and Transforms of Derivatives

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Terms and concepts. Inverse Laplace transform, Laplace transform of the derivative of a function.

The inverse Laplace transform. If  is the Laplace transform of , then  is the inverse Laplace transform of . This is written in notation as

  (1)

Unlike the Laplace transform, the inverse transform of a function of a function  cannot be determined through a process of differentiation or integration; it must be obtained by using a table of Laplace transforms in the reverse direction. Because it is not always possible to find a given function  in available tables, manipulations (that are often reminiscent of *u*-substitutions in integration) must often be carried out to “adjust” a function to match one of the entries in the table of Laplace transforms. Theorem 7.2.1 in the textbook [Zill] provides a brief table of inverse Laplace transforms, and Example 1 in the textbook demonstrates two different types of manipulations. Example 2 in the textbook illustrates how to break a complicated rational expression  into simpler fractions, and then use linearity properties (the inverse Laplace transform is still a linear transformation) plus adjustments to complete the determination of . Example 3 illustrates how to apply partial fractions to the computation of an inverse Laplace transform when  is a rational function whose denominator consists of a product of distinct linear factors.

Laplace transform of the first derivative of a function. Suppose that the Laplace transform of  is known. Is there a relatively easy way to compute the Laplace transform of the derivative (or, more generally, the Laplace transform of second or higher derivatives of )? The answer is “yes” to both questions. (Note that we have returned here to consideration of the “forward” Laplace transform, not the inverse Laplace transform.)

Let us start with the first derivative ; assuming that  is continuous on , its Laplace transform may be written using the definition from the previous section:

  (2)

Using integration by parts, with ,  and , , we obtain

  (3)

Evaluating the limits on the first term, and noting that the second term is simply *s* multiplied by the Laplace transform of , allows us to write

  (4)

Provided that  satisfies the “necessary and sufficient” condition from Theorem 7.1.2 of the textbook (which is likely, given that  exists), the limit  goes to zero for at least some values of *s*, yielding

  (5)

This last result is worthy of a place on your “cheat sheet,” written in whichever of the above two forms makes the most sense to you.

Laplace transform of the second derivative of a function. We continue now with the derivation of a formula for the Laplace transform of , which is again assumed continuous on . The technique is comparable to the method used for the first derivative, and uses the result of that calculation as a substitution in the last step:

  (6)

Letting ,  and , , we obtain

  (7)

Evaluating the limits on the first term, and noting that the second term is simply *s* multiplied by the Laplace transform of , allows us to write

  (8)

and some simplification gives

  (9)

Finally, substitution of the expression found in equation (5) for  gives

  (10)

  (11)

Laplace transform of the third and higher derivatives of a function. It should be clear that this process may be continued iteratively for higher derivatives of , provided they exist. The result obtained for the derivative of order  requires use of the transform previously found for the derivative of order *n*; therefore, a general formula for  may be shown to hold by using an inductive proof. This result is

  (12)

Application of the Laplace transform to solving ODEs. From the above results, we note that the Laplace transform of a derivative allows us to write the transform of any derivative  as an algebraic expression consisting of a multiple of the transform of  and a sum of constants. In other words, in the same sense that the auxiliary equation converts a second-order ODE with constant coefficients into a quadratic equation, the Laplace transform converts an IVP of arbitrary order (provided the equation has constant coefficients) into an algebra problem. If the inverse Laplace transform for the result can be determined (which can be problematic if it can’t be found in a table), we obtain a near-immediate solution to the IVP, usually without having to perform any integration.

It is traditional to write the Laplace transform of the solution  as . If the equation is nonhomogeneous, then the Laplace transform of the input function  (that is, the expression on the right side of the equals sign when the equation is written in standard form) is traditionally written as .

Example: First-order IVP with constant coefficients: Solve , provided .

This is a homogeneous equation, so we treat the input function as . As a preliminary, we may use the linearity property of the Laplace transform to show that . This is because (using a table of Laplace transforms)

  (13)

We start by taking the Laplace transform of both sides of the given ODE, and using the linearity property to simplify:

  (14)

Next, we substitute equation (5) for the transform of ,  for the transform of , and 0 for the transform of 0:

  (15)

The initial condition states that , so we substitute that detail and distribute the 2:

  (16)

We solve for  by collecting terms and applying some basic algebra:

  (17)

To complete the solution, we must find the inverse Laplace transform of . Consulting the brief table of inverse transforms on page 287 of the textbook, we see that the closest match to the above result is item (c), . We manipulate  so it resembles the expression inside the brackets of the inverse transform:

  (18)

Applying the inverse Laplace transform to both sides of this last equation gives

  (18)

so

  (18)

There are no arbitrary constants because the initial value information has already been incorporated into this result. So, this technique simultaneously solves the ODE *and* determines the value(s) of the arbitrary constant(s) at the same time! (The diligent student should compute , plug  and  into the original ODE to verify the solution, and check, by substituting  into the solution, that the initial condition is satisfied.)