MATH V23 LECTURE NOTES (Bowen)  
Section 7.1  
Definition of the Laplace Transform

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Terms and concepts. Transform, linearity, integral transform, convergence, divergence, Laplace transform.

Introduction. (Discussion of ordinary derivatives and indefinite integrals of a single variable as examples of transforms.) These can also be shown to be linear transforms, using the definition

 (1)

Integral transforms. *Definite* integrals of a function of *two* variables can also act as transforms. An example would be

 (Note: *y* is treated as a constant, due to the *dx*) (2)

which may be thought of in two ways; it either transforms a function of both *x* and *y* (in this case, ) into a function of *y* (), or (if the factor  is used consistently for *all* functions of *x*) transforms a function of *x* (in this case, ) into a function of *y* (again, ).

If the limits of integration happen to be  on a transform involving an integral, then the transform is called an ***integral transform***. Because integral transforms are, by definition, improper integrals, they should technically be solved by using limits, as was done in MATH V21B. (See equation (1) in the textbook [Zill] for an illustration of this.) If the limit exists, then the integral exists and is said to be ***convergent***; otherwise it is ***divergent***. Integral transforms are potentially useful in that they may be used to solve certain kinds of differential equations. We will see this in section 7.3 of the textbook, after we cover some additional preliminaries in section 7.2.

In multi-variable calculus, it is traditional to use *x* and *y* as the independent variables (for functions of exactly two variables). In the context of transforms, however, it is traditional to use *s* and *t* as the independent variables, where, by convention, *t* is the pre-transform independent variable, and *s* is the post-transform independent variable. Among the most well-known and useful of the integral transforms is the ***Laplace transform***, which transforms a function of *t* (generally written as  or ) into a function of *s*. The Laplace transform  is traditionally written as , and the Laplace transform of  is traditionally written as . Note the use of lower-case function names to represent the pre-transform function, and upper-case function names to represent the post-transform function. The definition of the Laplace transform is

 (3)

where the factor  is *always* used in the integrand to multiply the function  to be transformed. This transform is defined if and only if the integral on the right side of the above equation exists.

Examples 1 through 4 in the textbook show the computation of the Laplace transform for common functions such as , ,  (where *a* is a constant), and . Example 5 demonstrates the linearity of the Laplace transform, and theorem 7.1.1 provides a table summarizing the results of performing the Laplace transform on several common types of functions. Much more extensive tables of Laplace transforms have been tabulated, not unlike the tables of integrals introduced in MATH V21B. Integration by parts is frequently required to compute the Laplace transform of a given function; you may wish to review this technique if necessary to prepare you for the homework problems.

Sufficient conditions for the existence of the Laplace transform. Because the Laplace transform involves (by definition) an improper integral, the resulting limit may or may not exist. In turn, because this transform is only useful if it converges, it can be helpful to know whether it converges even before carrying out the necessary integration (which can be avoided if it is known in advance that the integral diverges). If a function  meets two specific conditions, then theorem 7.1.2 in the textbook guarantees that the Laplace transform exists, at least for certain values of *s*.

The conditions are that (1)  is piecewise continuous on , and (2) that  is of exponential order if *t* is greater than some finite constant *T*. ***Exponential order*** is defined in Definition 7.1.2 in the textbook. In ordinary language, if  is of exponential order, it means that, for large values of *t*, it increases no more rapidly than any increasing exponential function  would.

Many, but not all, common functions, have Laplace transforms; notable exceptions include  (which is not piecewise continuous on ), and  (which increases more rapidly than any exponential function of the form ). Example 6 in the textbook demonstrates how to find the Laplace transform of a piecewise function.

Finally, Theorem 7.1.3 asserts that if the two sufficient conditions described previously are met, then the Laplace transform  of a function  approaches a value of zero as *s* approaches infinity. In other words, the graph of  has a horizontal asymptote at zero, at least on the right-hand edge of the graph. It should be noted that not all Laplace transforms have this property, as some of the integrals converge even if the sufficient conditions are not met. In such cases, it is possible that the infinite limit of  has a value other than zero.

Example using direct integration. Find the Laplace transform of . Because this is an integration example, we use Definition 7.1.1 on page 279 of the textbook:

 (4)

As is often the case, the best way to compute this transform is to use the method of integration by parts. We may use the “LIATE” or “LIPTE” mnemonics to help us with the selection of appropriate *u* and *v* expressions. There are three factors: , , and , so we have an exponential function (“E”), an algebraic/polynomial function (“A” or “P”), and a trigonometric function (“T”). When there are more than two factors in the integrand, integration by parts usually works best if *u* is set equal to the product of all the factors except the last, and *dv* is set equal to the remaining factor. (Multiple uses of the technique may be required to provide sufficient simplification of the integral.) As the  factor (“E”) is the last letter in “LIATE”/“LIPTE,” we set that (along with the *dt*) to *dv* (that is, ), leaving . We find *du* from applying the product rule to , obtaining , and we find *v* by integrating *dv*, obtaining . So, the Laplace transform becomes

 (5)

Because *s* is considered a constant for the purposes of the above integral, we may factor out  from both parts of the expression for . We also split the remaining integral into two terms, giving

 (6)

We apply integration by parts again to the last integral above, with , , , and . This yields

 (7)

Again, factoring out  and breaking the last integral into two terms, we obtain

 (8)

Distributing the coefficient  through the braces on the second line of the above equation gives

 (9)

Excluding the coefficient of , the integral in the last term above is a repeat of the original integral for , so we solve the integration by replacing the integral with , and then algebraically isolating this quantity on the left side of the equation:

 (10)

 (11)

We factor the terms on the left side of the equation, and look up answers for the remaining integrals in a table of integrals (try integrals #29 and #30 from the inside front cover of the textbook), obtaining

 (12)

Multiplying both sides of this equation by  and applying the limits of integration (some details of using L’Hopital’s rule and subsequent simplification are omitted) gives

 (13)

Isolating  gives the result:

 (14)

Example using a table of Laplace transforms. As can be seen from the previous example, the computations of Laplace transforms using the definition can be both lengthy and tedious. Therefore, tables of Laplace transforms of various functions have been compiled, and have become widely available (originally in special-purpose books of tables, and, more recently, online). This can save considerable time and effort.

Let’s find . As a reminder, the definition of  in terms of exponential functions is

 (15)

Therefore

 (16)

Using the linearity property of the Laplace transform, we may factor out the , and split the remaining expression into two terms:

 (17)

Using formulas **(a)** and **(c)** from Theorem 7.1.1 on page 282 leads us quickly to the solution:

 (18)

 (19)