MATH V23 LECTURE NOTES (Bowen)  
Section 6.3  
Series Solutions About Singular Points

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Terms and concepts. Regular singular point, irregular singular point, method of Frobenius, index, indicial equation.

Introduction. In the previous section, we found solutions to the standard form of the homogeneous second-order equation

 (1)

that were power series; that is, series of the form

 (2)

where *c* was a constant value of *x* about which the Taylor series was expanded, and the coefficients  were determined through a recurrence relation. To ensure that a Taylor expansion existed, we required  and  be ***analytic*** (infinitely differentiable) at , in which case we said that  was an ordinary point.

In this section, we address the case in which either  or  is not analytic at ; that is,  is a ***singular point***. Singular points come in two flavors, ***regular singular points*** and ***irregular singular points***; these will be defined in more detail shortly. Equations containing irregular singular points have no reliable means of solution; this section will therefore focus on solutions near regular single points only. When seeking series solutions at regular singular points, it is possible that we may find only one series solution rather than two. In such cases, we may appeal to the method of reduction of order to find the second solution, as we studied in section 4.2 of the textbook [Zill], or use additional methods as suggested in this section.

Classification of singular points. Starting with equation (1), a ***regular singular point*** is any singular point  for which the expressions

 and  (3)

are both analytic at , even though  or  are not analytic. If a singular point fails to satisfy either of these conditions, then it is an ***irregular singular point***.

An alternative definition (not provided in the textbook, but it might make your homework easier) is that a regular singular point is a singular point  for which the two-sided limits

 and  (4)

both exist. If either limit fails to exist (in this context, a value of  constitutes failure to exist), then  is an irregular singular point. Depending on the functions  and , a single equation may have both regular and irregular singular points, or it may have no singular points at all.

Example: Finding and classifying singular points. Consider the second-order homogeneous ODE

 (5)

(If you are not completely comfortable with parentheses, the polynomial at the front of this equation is multiplying *only* the  term.) Before starting to find and test singular points, we note that this is not in standard form, so we divide both sides by the coefficient of the second-derivative term to obtain the standard equation

 (6)

which gives us

 and  (7)

By factoring either denominator above to obtain , we see that  and  are both undefined (therefore not differentiable, and therefore not analytic) at  and ; these are the singular points of the equation. (All other real values of *x* are regular points, whose solutions were addressed in the previous section. In an advanced course, we would also investigate whether  were singular points, if a solution valid for very large *x* were desired, but that is not covered in this course.)

Let us test the singular point  to determine whether it is regular or irregular. If we use the textbook’s definition of regular, we build and test the expressions designated in equation (3):

 (8)

and



To complete the test of , we evaluate  and . If both are defined, they are analytic at , and the point is regular; if either is undefined, the undefined one is not analytic, and the point is irregular. We find

 (9)

 (10)

Both  and  are defined, so  is a regular singular point.

If we use the alternative definition (the one involving limits) we obtain (starting with equations (4))

 (11)

and

 (12)

Since both  and  exist, we again identify  as a regular singular point.

We now test the other singular point . Using the limit method gives

 (13)

and

 (14)

The limit  does not exist, so we conclude that  is an irregular singular point. The conscientious student should independently verify that the textbook’s definition of regular singular point is also *not* satisfied for . If we had been paying close attention, we could have skipped the test of , because , being undefined, already disqualified  as a regular singular point.

Solution about a regular singular point: method of Frobenius. This method of solution is similar in many respects to the solution about an ordinary point; we set up a generic series to represent the solution *y*, compute the first and second derivatives of the series to represent  and , and substitute these into the equation to determine the coefficients of each term of the solution. Theorem 6.3.1 (by Frobenius) on page 254 of the textbook gives the form of the series solution as

 (15)

which can also be abbreviated as

 (16)

if we bring the leading factor inside the sum, and apply exponent properties. The Frobenius solution (above) comes with some caveats; namely:

1. In addition to finding the coefficients , we must find the value of the exponent *r* on the factor in front of the summation, which will generally be a real constant. (Details of how to find *r* will be coming shortly…and there is a shortcut.)
2. If *r* is either negative or a fraction, then the resulting series no longer qualifies as a power series (although it likely still represents a solution). To be safe, we will start calling the solution a “series” rather than a “power series,” just in case this happens.
3. The method might obtain two separate series (that is, two fundamental solutions) for us, but it guarantees us only one. If it only found one, then other methods would be needed to find the second fundamental solution. (The second solution is still out there; it’s just that the Frobenius method wouldn’t help us find it). Example 4 on page 258 of the textbook explicitly deals with that case.
4. We cannot be assured of finding a solution *at* the regular singular point , only *near* it. If we plug in  into the Frobenius solution above, we obtain either  (for nonnegative *r*) or  (for negative *r*) as the sum of the series, so even if there were a solution *at* , the above summation formula would not correctly represent it. However, the Frobenius theorem also guarantees us that there will always be at least a small (that is, nonzero) radius of convergence about  for the series solution(s) we obtain; just keep in mind that “nonzero” might mean “one-billionth.” So, in some (possibly very small) neighborhood of *x* values located near , we are guaranteed that our series solution(s) will be valid (that is, it will converge and represent a solution to the ODE).
5. The method is efficient only if  is the regular singular point. The calculations become nearly intractable for other values of *x*. If you encounter an equation for which you wish to expand about a nonzero regular singular point, it is generally worth the effort to shift everything horizontally by inventing a new variable  (where *h* is the value of the regular singular point) and plugging it into the equation so that the regular singular point of the shifted equation is .

Example: Applying the method of Frobenius. We illustrate the method by using it to solve the equation

 (17)

We rewrite this in standard form to begin the identification of singular points:

 (18)

We identify

 and  (19)

from which it is clear that  is a singular point (it makes both denominators zero). We use the limit-based definition of “regular singular point” to determine whether  is regular:

 (20)

and

 (21)

Both  and  exist, so  is a regular singular point.

We differentiate both sides of the general Frobenius solution (equation (16)) twice, using the power rule, and treating *n* and *r* as constants (they are!) to prepare to insert the necessary quantities into equation (17), obtaining

 (22)

 (23)

Note that it is not necessary to expand the series to differentiate; all you need to do is apply the power rule for differentiation to the general term. However, you may wish to verify term-by-term differentiation the first few times until you develop trust in this method.

Inserting the above two equations and equation (16) into equation (17) (which is not in standard form, but using it helps us avoid fractions), and explicitly replacing *c* with 0, yields

 (24)

We move the coefficients of the first two sums to the inside of each sum, and distribute the expression in front of the third sum to expand it into two new sums (one for each term in the coefficient):

 (25)

We use exponent properties to consolidate the powers of *x*:

 (26)

We factor out  from each term to simplify the exponents:

 (27)

We perform a *k*-substitution in the last sum to make all the powers of *x* equal to *n*. Note that this changes the limits of summation:

 (28)

We explicitly write out the details of the  term from each of the first three sums; this allows us to rewrite these sums with a starting index of , so that all four sums contain both the same powers of *x* and the same limits of summation:

 (29)

Some simplification yields

 (30)

At this point, all the powers of *x* that are 1 or greater (excluding the  in front of the large braces) are inside summation symbols, and all the  terms are explicitly listed in front.

Now we begin to match coefficients on the left and right sides of the equation. The constant terms (the ones that contain ) at the beginning of the above equation must add to zero to match the corresponding coefficient of zero on the right side of the equation, leading to

 (31)

The coefficient , as in the case of the ordinary point, may be set arbitrarily; for simplicity, let’s pick . The equation becomes

 (32)

This quadratic equation (if we solve it) will tell us what to use for *r*! Because the exponent *r* on the leading factor  is called an index, this quadratic equation is called the ***indicial equation***. It is a regular feature of Frobenius problems (which is a good thing; otherwise we couldn’t obtain a correct series).

Fortunately, there is a shortcut that allows us to obtain this equation in advance, as soon as we’ve identified the regular singular point. The indicial equation *always* has the form

 (33)

If the coefficients  and  look familiar, it’s because we’ve already seen them before, in equations (20) and (21). They were the values of the limits we used to test whether  was a regular singular point! We found that  and ; to illustrate the shortcut, we may insert these into the above formula, giving

 (34)

which simplifies to

 (35)

Multiplying both sides of this last equation by 2 gives the same indicial equation we obtained in equation (32), but with substantially less effort. Either of these equations is readily solved by factoring; the solutions are  or  are the solutions. Each solution for *r* will give rise to a different set of recurrence relations for the coefficients . To distinguish between the coefficients for the  solution and the , we will name the coefficients for the  as , and rename the coefficients for the  solution as .

To obtain the  and  for , we must extract the appropriate terms from the summations in equation (30). We start with  and ; these are the coefficients of the  () terms in all the sums in equation (30), and are obtained from the  terms in each sum. We find (for )

 (36)

and (for )

 (37)

Next up are  and  (the coefficients of the  terms, obtained from the  terms in each sum in equation (30)). For :

 (38)

The value of  (for , again obtained from the  terms in each sum) is

 (39)

For  and , we extract the  coefficients from all sums in equation (30); for ,

 (40)

and, for ,

 (41)

This process may be continued indefinitely to obtain as many coefficients as desired.

Keeping in mind that any series solution is necessarily approximate (the full solution would involve all the infinite number of terms in the series developed, whereas we can only write a finite number of these), we write the first few terms of each power series solution to demonstrate how the above coefficients are used. Plugging our results for , , , , , , , , and  into equation (16) gives the approximate solutions  and , where  is used for , and  is used for :

 (42)

 (43)

A simple way to write the fundamental solutions is to set  and ; the fundamental set then becomes

 (44)