MATH V23 LECTURE NOTES (Bowen)  
Section 6.2  
Series Solutions About Ordinary Points

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Terms and concepts. Radicand, “root” of radical, integrand, summand, limits of summation, re-indexing by *k*-substitution, piecewise summation, analytic function, ordinary and singular points, coefficient matching, recurrence relation.

Introduction. We seek to solve second-order homogeneous ODEs by finding a power series (in general, a Taylor series) that represents the solution *y*. (Since the equation is homogeneous, we do not need to find a particular solution .) More specifically, we seek a solution of the form

 (1)

(where *c* is a constant value of *x* about which the Taylor series is expanded) to the equation

 (2)

Note that, in this context,  and  are functions, not merely constant coefficients, as they have been through chapter 4 of the textbook [Zill].

A complete characterization of the solution requires us to find expressions for each of the coefficients  in the series of equation (1), and to identify two independent solutions (that is, a fundamental set). (Since there is an infinite number of coefficients, we will try to find a general expression for them.) In this section, we will further require that  and  be ***analytic*** at , which is to say that they are infinitely differentiable at , thus guaranteeing that each has a Taylor series around . If  and  are analytic at , we call  an ***ordinary point***. If either  or  is not analytic at , then we call  a ***singular point***. The method of solution is different for singular points; we will address this when we reach section 6.3 of the textbook.

Differentiation of power series. To begin solving equation (2), we insert the generic solution *y* from equation (1). To accomplish this, we must also find expressions for  and . In your calculus course, you were likely shown that a convergent power series can be differentiated or integrated term-by-term. To see what we obtain for  and , it is instructive to expand equation (1), obtain its first and second derivatives, and then look for a pattern that allows us to rewrite  and  in the more compact sigma (summation) notation. Equation (1) becomes

 (3)

and term-by-term differentiation yields

 (4)

 (5)

Each of the above series has a pattern relating the coefficients, the subscripts, and the powers of ; with a bit of thought, we may convert these to summation notation:

 (6)

 (7)

When we begin to solve our equation, we will find it more convenient to write these sums so that each has the same power of . We accomplish this via a “*k*-substitution,” which, like a *u*-substitution in integration, requires us to change both the variable and the limits (in this case, the limits of summation rather than the limits of integration). In equation (6), we let  (so ) to obtain the alternative form

 (8)

and in equation (7) we let  (so ) to obtain

 (9)

In general, whenever a *k*-substitution is needed to re-index a sum, the value selected for *k* should be the power to which  is raised. If that power is already just *n*, then no *k*-substitution should be required.

Going back to calculus one more time, note that when we integrate, the independent variable is called a “dummy variable”, because it doesn’t really matter whether we integrate (for example)  or ; we get the same answer regardless of whether the variable is *x* or *t*. Summations are analogous; it doesn’t matter whether the index is *k* or *n*. So, we may rewrite equations (8) and (9) as

 (10)

 (11)

You may wish to write equations (1), (10), and (11) on your cheat sheet for the next exam, rather than having to re-derive them for every problem (they will always be the same). You would do well to be skeptical about equations (10) and (11) until you become more familiar with *k*-substitutions; verify them by plugging in the first few values of *n* to ensure that you still obtain the corresponding expansions from equations (4) and (5), respectively.

After we plug the series for , , and  into the differential equation, the next steps are to perform *k*-substitutions as needed so that each summation contains the same power of  or , and then make additional adjustments so that all the series have the same limits. Next, we consolidate the sums into a single sum (to the extent that this is possible; there may be a small number of additional terms in front of the infinite sum). Finally, we set each of the coefficients in the sum on the left side of the equation equal to zero (since the right side of the equation is identically zero) to determine expressions for the . This completes the solution. What we usually find is that each of the  has a value that depends on an arbitrary selection of values for ;  and ; or , , and .

Example. Consider the equation

 (12)

and find a series solution expanded about  (that is, a Maclaurin series for *y*). We note that  and , so any value of *c* (including ) is an ordinary point, as these functions are readily (and infinitely) differentiable. Plugging in equations (1) and (11), and setting , give

 (13)

Distributing the *x* in front of the second sum into the ***summand*** (the expression to the right of the sigma) gives

 (14)

where we have treated  as  and used exponent properties to consolidate the product . Performing a *k*-substitution on the second sum (with ), and reverting the dummy variable back to *n* afterward, give

 (15)

which serves the purpose of achieving identical values of the exponent in both sums. It is only possible to make the limits of summation agree by separating out the  term of the first sum:

 (16)

With matching limits and powers of x on the remaining sums, we may consolidate these:

 (17)

Because the zero on the right side of the equation may be thought of as

 (18)

we set each coefficient of  on the left side of the equation equal to the corresponding coefficient of  (zero!) on the right side of the equation to obtain the solution. For specific values of n starting with zero, we obtain

 (19)

Solving the first row in equation (19) gives

 (20)

From the remaining rows, we see that the value of  depends on the value of , the value of depends on the value of , the value of  depends on the value of  (which we just found to be zero), and so on. This dependence of later coefficients (those with larger subscripts) on the values of earlier coefficients (those with smaller subscripts) is a general feature of the method of series solutions, and the general form (which is called a ***recurrence relation***) is shown in the last row of equation (19). Note that there are no formulas for  or ; these may be set arbitrarily, and their roles are equivalent to those of the constants of integration  and  from the complementary solution

 (21)

that we found in chapter 4 of the textbook. We now seek a pattern for the values of all the coefficients, which should enable us (if we find a pattern; in most problems there is no pattern!) to write a series for the solution *y*. Solving each row in equation (19) gives

 (22)

So, there is a pattern, but not a particularly nice one. (For more complicated results, it may only be possible to write the first few terms of the series expansion, rather than the general term. This can still lead to a good approximate solution, provided that the series converges quickly.)

Every third coefficient is zero, starting with , so , , , *etc.* If we assume that , we may write this as .

Starting with , every third coefficient depends on the arbitrary value we assign to . The general form for these coefficients (this time assuming that ) is

 (23)

Finally, starting with , every third coefficient depends on the arbitrary value we assign to . The general form for these coefficients (with ) is

 (24)

Plugging all these coefficient formulas into equation (3) (again, with ) gives

 (25)

or, after eliminating the zero terms,

 (26)

We may break this solution into one summation for each element of the fundamental set by factoring out  or  from each term that contains these quantities:

 (27)

or (using summation notation for all terms to which it can be applied)

 (28)

where each quantity in brackets is an element of the fundamental set, and  and  are arbitrary constants. (In an IVP, it would, in principle, be possible to obtain specific values for  and ; in practice, this may be impossible unless the infinite series can be evaluated or closely approximated for the given values of  and .)

It should be noted that if the problem were modified to request an expansion around a non-zero value (for example, ), none of the above work could be re-used. It would be necessary to start the entire problem from scratch, and a completely different series would be obtained.

If an obvious pattern for the  cannot be obtained from the recurrence relation (making it impossible to write the series using summation notation), it is generally considered acceptable to write the first few terms of the series for *y* without including the general term (the one with *k* or *n* in it). In such cases, at least *four* nonzero terms should be provided. Also, the value of *c* should be chosen at or near the value of *x* for which a solution is desired, so that the series obtained for *y* converges rapidly. This allows a good approximation to be found even though only a small number of terms may be used. If  is large, there is also a risk that the series for *y* will not converge, rendering one or both of them useless as a solution.