MATH V23 LECTURE NOTES (Bowen)  
Section 4.6  
Variation of Parameters

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Introduction. In section 4.4 of the textbook [Zill], we solved nonhomogeneous linear equations with constant coefficients. Although this is an important and useful class of equations, there are many equations that either have functions as coefficients, or for which the input function  does not fall into one of the categories addressed in section 4.4. Variation of parameters is a technique that allows us to solve a broader variety of second- and higher-order linear differential equations.

Motivation. We start by reviewing the solutions we’ve already discovered for first-order linear equations (see section 2.3 of the textbook), but reframing them in terms of the theory developed in section 4.1 for higher-order linear equations. Perhaps the most important concept arising from that theory is that the general solution of a nonhomogeneous linear equation is composed of the sum of two pieces: the complementary function  (the solution to the corresponding homogeneous equation), and any particular solution  of the nonhomogeneous equation.

For the *homogeneous* first-order equation

 (1)

we expect to obtain a one-parameter family of solutions ; by employing an integrating factor, we find that such a solution is given by

 (2)

which we may readily verify, because

 (3)

and plugging the preceding two equations into equation (1) results in an identity:

 (4)

which satisfies our standard of proof for verifying a solution.

Moving to the *nonhomogeneous* first-order equation

 (5)

the theory from section 4.1 of the textbook assures us that if we can find *any* particular solution , then the general solution will be

 (6)

where  is the complementary function we just found in equation (2). When we first sought a solution for this in section 2.3, we found that multiplying both sides of this equation by an integrating factor  made Equation (5) integrable, because the left side became the derivative of a product of functions:

 (7)

 (8)

Integrating both sides of this equation gave us

 (9)

where  was the constant of integration obtained after finding the antiderivative in the other term on the right side of the equation. To complete the solution, we isolated *y* on the left side of the equation by dividing out the exponential factor, leading to

 (10)

In comparing equation (10) with equation (2), we notice something new; namely, that the integrating factor method used to solve the first-order nonhomogeneous equation *automatically* generated a term  that exactly matched the complementary function . We conclude (by also comparing equation (10) with equation (6)) that the last term on the right side of equation (10) must be a particular solution of equation (5); that is,

 (11)

We also notice an interesting relationship between  and . Let us take the expression  and denote it as , so

 (12)

By taking the reciprocals of both sides, we may also write

 (13)

We then note that  is a key component of both  and ; namely, by substituting equation (12) into equation (2), we see that

 (14)

and, by substituting equations (12) and (13) into equation (11), we see that

 (15)

If, in the preceding equation, we replace  with  for brevity, so that , we discover a possible model for finding solutions of higher-order equations; namely, finding the complementary function, removing the constant of integration, and then multiplying by an unknown function  (used in a manner similar to an integrating factor) in hopes of finding a particular solution of the form . The “parameter” that we are varying when we employ the method of variation of parameters is the function .

For second-order equations, the complementary function has two terms , so our guess for the particular solution has the form

 (16)

From this, it should be clear that the extension to an equation of order n would be

 (17)

Note that if this method is used, it is essential in every case to obtain, as the first step, the complementary function (that is, the general solution to the corresponding homogeneous equation). Because the set  is a fundamental set, we may also safely assume that their Wronskian is nonzero: . This will become an important thing to be aware of in just a bit, as we continue to develop this method of solution.

Rather than beginning our effort by attacking equations of every order, let’s find a particular solution for the second-order equation

 (18)

(If the leading term  has a coefficient function other than 1, then simply divide the entire equation by this function before starting.) We assume that the fundamental set of solutions  to the corresponding homogeneous equation has already been found. To plug the proposed particular solution  into equation (18), we’ll need its derivatives. The first derivative is found using the product rule on each term:

 (19)

From this result, we can see that the second derivative will have quite a few terms. To simplify it, we are going to assume that two of the four terms in the first derivative above sum to zero, and therefore vanish. This assumption is completely *ad hoc*, and unwarranted by anything we already know about the equation or its solution, but it makes the derivative computations easier, *and* turns out not to affect the answer (for reasons that are beyond the scope of this course; naturally, if we had any doubt about the correctness of the solutions we obtained by making this assumption, we could always check them by direct substitution into equation (18)). The specific assumption is that

 (20)

(a result that will also be used later), which happily shortens the expression for the first derivative to

 (21)

The second derivative then becomes

 (22)

Substituting the preceding results into equation (18) initially yields

 (23)

but distributing, rearranging of the terms, and factoring out  and  give

 (24)

The second and third pairs of brackets in this expression (underlined and shown in red) both go to zero. This is because  and  are, by construction, solutions to the homogeneous form of equation (18), and the expressions contained within the brackets represent the left side of the homogeneous equation with one or the other of these two known solutions plugged in.

It is not possible to solve the resulting equation

 (25)

without some assistance, because there are effectively two “unknowns”  and , but only one equation (we need one more equation). We appeal to our earlier assumption from equation (20) to serve as the needed second equation, giving us our very first *system* of linear equations to solve! As with algebraic systems of equations, the method of substitution works well; we isolate the term  from equation (20) to give

 (26)

and substitute this into equation (25), yielding

 (27)

The only unknown quantity in this equation is , so (with a bit of algebra) we isolate it:

 (factoring out ) (28)

 (creating LCD) (29)

 (consolidating numerators) (30)

 (isolating ) (31)

We may insert this result back into equation (26) to obtain the other unknown :

 (isolating ) (32)

In the preceding two equations, we note that each denominator on the right side is identical; moreover, each is equal to the Wronskian of  and . Because  and  are fundamental solutions of the homogeneous equation, we may be assured (as was noted previously) that the Wronskian is nonzero, giving us greater assurance that these solutions exist, at least on the interval(s) where , , and  are continuous. So, we may shorten these solutions to

 and  (33)

We integrate these results to obtain  and  (the constants of integration may be assumed zero, because including them would create additional terms that solved only the homogeneous equation; that is, they would not create anything that would contribute to the generality of the particular solution we seek). The results are

 and  (34)

This bring us to the end of the derivation! By substituting these results into equation (16), we obtain the particular solution  of the second-order nonhomogeneous equation:

 (35)

The general solution of the second-order nonhomogeneous equation (including the complementary function required by equation (6)) is therefore

 (36)

This method may be extended to third- and higher-order equations, but doing so would be beyond the scope of this course. Let’s try some examples.

Example 1: Let’s solve the second-order equation

 (37)

Although the left side of the equation is not very fancy (it still has constant coefficients, just as in textbook section 4.4), we could not solve it using the methods of section 4.4, because the input function  on the right side of the equation does not match one of the forms addressed in that section. With the method of variation of parameters, the essential first step is to obtain the complementary solution (in this case, by solving the auxiliary equation). Because the auxiliary equation in this case has repeated real roots , we have to use the “multiply by *x*” approach, and obtain

 (38)

as the complementary function, so  and .

The second step, in general, is to find the Wronskian of these two functions, which is

 (39)

The third step is to obtain the particular solution from equation (35):

 (40)

The first integral on the right is readily solved with the substitution , , and the integrand of the second integral on the right is the derivative of . So, the integrals are tractable, and we obtain

 (41)

where we justify removing the absolute values by noting that  is inherently positive, and we omit the constants of integration for the reasons stated in the paragraph following equation (33). The general solution, obtained by reintroducing the complementary function, is therefore

 (42)

Example 2: Now let’s try to solve

 (43)

given that  and  are solutions to the homogeneous equation.

This is our first second-order equation without constant coefficients; because we don’t yet know how to solve the corresponding homogeneous equation, we have to ask for the fundamental solutions to be provided; otherwise we have no way to go forward. As a precaution (an exercise to be left to the student), we should verify (by direct substitution) that  and  do solve the homogeneous equation, as claimed, and also that they are independent. To confirm the latter, we investigate the Wronskian, and find that

 (44)

which is nonzero provided that the interval of validity excludes . We can justify this by noting that we cannot place the original problem into the required form unless we first divide the equation by , which would be impossible if *x* were allowed to take on the value 0. Assuming that *x* is nonzero allows us to rewrite the original problem as

 (45)

With the information we now have, we are ready to find the particular solution from equation (35), which is

 (46)

The general solution, obtained by reintroducing the complementary function, is therefore

 (47)

Remember that, although the general solution is defined and continuous at , any interval of validity must exclude this value, for the reasons noted above.