MATH V23 LECTURE NOTES (Bowen)
Section 4.4
Undetermined Coefficients—Superposition Approach

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Introduction. In section 4.3 of the textbook [Zill], we solved homogeneous linear equations with constant coefficients. In this section, we will expand on this by learning to solve certain nonhomogeneous linear equations with constant coefficients. The method of undetermined coefficients relies on the fact that the derivatives of certain classes of functions (constant functions, polynomial functions, exponential functions, sine/cosine functions, products of exponential with sine/cosine functions, and products of polynomial functions with any of the preceding function types) are also functions of the same class. For an equation of the form

  (1)

we will make an educated guess that if the ***input function***  is a sum of functions of one or more of these types, then a particular solution  will also be a linear combination of the same types of functions; the key solution-finding step will be to determine which specific linear combination is correct by calculating proper coefficients. For example, if , we will guess a particular solution of the form , then strive to find the correct coefficients *A* through *F*. (The cosine term is added because the various derivatives of  may be either cosine or sine functions, and the polynomial terms are added to provide a complete polynomial of the same degree as the one appearing in , not unlike what is required for missing terms in polynomial long division.) Table 4.4.1 on page 146 of the textbook suggests trial forms of  corresponding to selected examples of input functions .

As a reminder, the theory of linear equations developed in section 4.1 states that the general solution of the nonhomogeneous problem (equation (1)) is the sum of the general solution of the corresponding homogeneous equation (also known as the ***complementary function*** ), and any particular solution  of the nonhomogeneous equation; that is,

  (2)

So, to solve equation (1), we will have to use the methods of section 4.3 to find , *and* the methods of this section to find , before we will be in a position to construct the complete general solution .

Occasionally a glitch arises in this method. If a guess for  from Table 4.4.1 includes a duplicate of a term already appearing in the complementary function , then we multiply that guess successively by , , , or a higher power  to create new guesses, until we obtain a guess that no longer appears in the expression for . The power of *x* used should be the smallest one that allows us to avoid having duplicate functions appear in both  and . With no further ado, we are ready to try some examples.

One other thing to notice is that the number of undetermined coefficients should be kept to a minimum. For example, if , we might initially guess . However, if we distributed this expression, we would find that each term had two coefficients: , which would be redundant. So, we’d remove the coefficient *D* and guess  instead, to reduce the number of coefficients from 4 to 3. Constants that multiply the entire function  may also be ignored; for example, if , then the leading 5 should not be included in 

Type I: (no duplication between the complementary function and the guess for the particular solution). Consider the example

  (3)

for which we would like to find the most general solution. It is nearly always advantageous to begin by finding the complementary function. So, we set up the corresponding homogeneous equation

  (4)

and solve it readily by nothing that the auxiliary equation  easily factors to , from which we obtain quadratic solutions  and . These, in turn, lead directly to the complementary function .

Turning now to the particular solution, the term  in  suggests including terms  in our guess for the particular solution. The term  is a first-degree polynomial (*x*) multiplied by an exponential function, so we also insert a fully-formed first-degree polynomial  multiplied by a generic exponential function . However, when we write these last two and multiply them out, we obtain , which has a redundant constant *E* in front of each term. Eliminating the extra constant allows us to write our guess for the particular solution as

  (5)

To test our guess, we compute its first two derivatives and substitute them into the nonhomogeneous equation (3), in hopes that suitable constants may be found. We obtain

  (6)

  (7)

We plug our guess for  and its derivatives into equation (3) to calculate the values of the unknown coefficients *A*, *B*, *C*, and *D*:

  (8)

Factoring the above to separate the coefficients of each function in the “guess” gives

  (9)

Combining like terms, and setting coefficients equal, gives rise to a system of linear equations, in which the undetermined coefficients are the variables. The technique is not unlike the one used for partial fractions in integration:

  (10)

The solution to the above system is readily obtained from the method of substitution, starting with the second and fourth equations above, which end in zero, and the third equation, which immediately gives the value of the coefficient *C*. We obtain

  (11)

so, the solution for the particular solution  follows immediately by inserting these results into equation (5). We obtain

  (12)

Finally, equation (2) directs us to the general solution, which is

  (13)

The student should verify this solution by direct substitution into equation (3).

Type II: (duplication between the complementary function and the guess for the particular solution). Consider the example

  (14)

which, at first glance, seems very similar to the Type I example above, except for a bit of tweaking of the constant coefficients on the left side of the equation. The auxiliary equation for the homogeneous version of the equation becomes , which factors as , yielding  but with a multiplicity of 2. A review of Case 2 in the [Section 4.3 lecture notes](http://academic.venturacollege.edu/mbowen/courses/handouts/h_v23_lecnotes_4.3.docx) assures us that the complementary function is

  (15)

Turning now to the particular solution, the term  in  again suggests including terms  in our guess for the particular solution. The term , however, reminds us of the second term of the complementary function, which is simply a constant multiple of ; if we were to use this in our guess, it would not work (it solves the homogeneous equation, not the nonhomogeneous equation). In the Type I example above, our guess for this portion of  was ; to fix the problem of duplicate terms in  and , the standard procedure is to multiply this entire portion of the guess by *x*, giving a second guess of . However, this second guess also contains a term of the form , so it is still not satisfactory, and we multiply by *x* again to obtain a third guess of . This guess contains no terms of the form , so this is what we will use. The complete guess for the particular solution becomes

  (16)

We again find the derivatives of  to plug them into the nonhomogeneous equation (14).

  (17)

  (18)

Consolidating the like terms visible in the expression for the second derivative gives the slightly shorter expression

  (19)

We plug our guess for  and its derivatives into equation (14) to calculate the values of the unknown coefficients *A*, *B*, *C*, and *D*:

  (20)

Factoring the above to separate the coefficients of each function in the “guess,” and consolidating like terms, gives

  (21)

  (22)

  (23)

Setting coefficients equal gives rise to a system of linear equations, in which the undetermined coefficients are the variables.

  (24)

From from the third equation, we obtain , and from the fourth equation, we obtain . We multiply the first equation by 4, and the second equation by 3, then add to eliminate the *A* terms, giving  and therefore . Inserting this into the first equation gives , or , or . To summarize,

  (25)

so, the solution for the particular solution  follows immediately by inserting these results into equation (16). We obtain

  (26)

Finally, equation (2) directs us to the general solution, which is

  (27)

Examples 10 and 11 on page 149 of the textbook illustrate solutions for higher-order problems; note that the principles are very similar to those used for second-order equation in these two examples. These techniques may be used for solving problems 21, 23, 25, and 35 in the homework assignment.