MATH V23 LECTURE NOTES (Bowen)
Section 4.3
Homogeneous Linear Equations with Constant Coefficients

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

For the purposes of this section, a ***complex number*** is a number having the form , where *a* (the ***real part*** of *z*) and *b* (the ***imaginary part*** of *z*) are real constants, and  (or ). A ***pure imaginary number*** (or just ***imaginary number***) is a complex number whose real part is equal to zero. A ***real number*** may be thought of as a complex number  whose imaginary part is equal to zero. Note that the imaginary part of a complex number is defined as just the coefficient *b*, not the entire imaginary term *bi*. The ***complex conjugate*** (or just ***conjugate***) of a complex number is the complex number obtained by reversing the sign of the imaginary part; that is, the conjugate of  is , and vice-versa. Every *real* number is equal to its own complex conjugate.

Introduction. Homogeneous linear equations with constant coefficients are a special yet common instance of ODEs in which the coefficient functions multiplying the derivatives on the left side of the equation are all constants. Second-order equations of this type model common physical systems such as undriven simple harmonic oscillators (with and without friction) and undriven *LC* and *LRC* circuits (that is, circuits in which the capacitor is initially charged, then allowed to discharge naturally through the circuit, without an external power supply being present). The corresponding nonhomogeneous equations, to be discussed later, model driven systems (*e.g.*, “pumped” oscillators, or *LRC* circuits driven by an external power supply). In most cases, two independent solutions are readily obtained. However, in some special cases, only a single solution is initially obtained. In these cases, the technique of Reduction of Order, which we covered in section 4.2 of the textbook [Zill], may be applied to recover a second independent solution from the one first found.

In accordance with the theory of linear homogeneous equations developed in textbook section 4.1, the general solution to the second-order equation consists of all linear combinations of the two independent solutions obtained. At its simplest, the second-order constant-coefficient technique is often little more difficult than solving a quadratic equation. Higher orders require solving polynomial equations whose degree (and number of independent solutions) is the same as the order of the ODE, so you may wish to review synthetic division and the rational root theorem from your precalculus course (see Example 5 on page 139 of the textbook if you do not have access to your precalculus textbook) before tackling these. The solutions are typically a linear combination of either exponential, sine, or cosine functions (or products thereof, such as ), depending on the numerical values of the coefficients. This is because the exponential, sine, and cosine functions are the only elementary functions whose first, second, or higher-order derivatives can be constant multiples of the original function.

The auxiliary equation. The most general linear homogeneous ODE of order 2 is

  (1)

but for the purposes of this section, all the coefficient functions , , and  will be assumed constant. Changing to prime notation, and dropping the redundant functional dependence on *x*, allows us to rewrite this as

  (2)

where *a*, *b*, and *c* are all constants. Because exponential functions of the form , where *m* is a constant, have the special property that first and higher derivatives are constant multiples of the original function, we make an educated guess that the independent solutions might be of the form  and , and seek specific values of  and  that make the equation true (that is, they yield an identity when we substitute them and simplify). The general solution is constructed as the linear combination . For an IVP and certain BVPs, it is possible to determine specific values of the coefficients (parameters)  and  by fitting the general solution to the initial-value or boundary-value data after the general solution is found.

Let’s write either of the independent solutions as simply ; using the chain rule gives

  and  (3)

and we may substitute these directly into equation (2) as our first step toward the solution:

  (4)

Each term contains a factor , which is safe to factor out and then divide through on both sides of the equation, as the result of evaluating an exponential function is strictly nonzero. (We must always be careful not to inadvertently divide an equation by zero whenever the divisor contains a variable; otherwise we can “prove” almost anything, such as .) This leads to

  (5)

The equation on the right above, which is quadratic in *m*, is called the ***auxiliary equation***. Solving it, by factoring, completing the square, or (when necessary) using the quadratic formula, allows us to determine the values of  and  to use in stating the two independent solutions of the second-order equation. However, if a repeated solution is found to the auxiliary equation (when the discriminant , leading to ), then this quadratic-based method only provides one independent solution. Fortunately, we may resort to the Reduction of Order technique from section 4.2 to find a second independent solution (more on this below). Therefore, even when the auxiliary equation has a repeated root, it is always possible to find exactly two independent solutions of the second-order ODE.

For higher orders of ODEs, it is easy to construct the auxiliary equation directly from the original ODE. Simply replace  with 1,  with ,  with ,  with , and so on.

Characterizing the solutions. Just as the solutions to a quadratic equation may be classified according to the value of the discriminant , the solutions to a second-order homogenous linear ODE with constant coefficients depend very much on the value of this quantity, giving rise to several different types of behavior if the ODE models a physical system.

Case 1:  (auxiliary equation has two distinct real roots). Consider the example . The auxiliary equation is , and we may verify

  (6)

The auxiliary equation is factorable (use factoring when possible; it’s usually faster than the quadratic formula), giving , with solutions  and . The corresponding independent solutions to the ODE are  and , valid on . The general solution is the set of all linear combinations of these, or . In general, if the discriminant is positive, then the general solution will be the sum of two exponential functions. An exception would be the situation when  in equation (2), for which the general solution would be  (you should verify this as an exercise so you may see for yourself why the second term does not contain an exponential function).

An interesting result occurs when  and  in equation (2). The discriminant is still positive, and the independent solutions are  and , where both values of *m* are the same except for the sign (verify this using the equation  as an example). Consider the hyperbolic sine (sinh) and hyperbolic cosine (cosh) functions, whose definitions are

  and  (7)

(See section 3.11 of the calculus textbook [Stewart] for a more thorough discussion of these functions, their properties, and their derivatives.) By adding and subtracting these definitions to/from each other, and replacing each instance of *x* with *mx*, we find that

  and  (8)

so, the general solution may be written

  (9)

where *A* and *B* are constants (in an IVP, these would be uniquely determined by the initial conditions). That is, for this special case, the general solution may be written as either a sum of exponential functions, or as the sum of sinh and cosh functions (see the discussion leading up to textbook equation (11)).

Case 2:  (auxiliary equation has two repeated real roots). Consider the example

  (10)

The auxiliary equation is , and we may verify

  (11)

The left side of the auxiliary equation is therefore a perfect square, giving , with solution . The corresponding solution to the ODE is . Because this method does not provide a second distinct solution, we apply the method of Reduction of Order by defining , or, omitting the explicit functional dependence, . The first and second derivatives of  are (from the product and chain rules)

  and  (12)

so, we plug these into equation (10) to begin our search for the second solution:

  (13)

Here, we divided both sides by the nonzero quantity , and combined like terms inside the parentheses, to obtain the surprisingly simple last line. Integrating twice successively to obtain an expression for *u*, and restoring the explicit functional dependence on *x*, we find that

  (14)

from which it immediately follows that

  (15)

In this last expression, we see that the term  is a constant multiple of the previously known independent solution , so we ignore this term (and the constant ) as we construct the second independent solution. After these adjustments, the second independent solution becomes . In general, for a second-order equation, if the discriminant is zero and the first independent solution is , then the second independent solution will be , The general solution for this example becomes

  (16)

where specific values for the constants  and  could be found if initial conditions were also provided.

Case 3:  and  (auxiliary equation has two conjugate pure imaginary roots). If we return to the “interesting result” in Case 1 and tweak the example slightly so the equation reads

  (17)

then the auxiliary equation becomes  or . Taking the square root of both sides yields , suggesting the independent solutions  and . What do we do with the complex exponents? Does it mean there is no solution in this case? Not at all! A clever Swiss mathematician, Leonhard Euler (1707–1783), proved the simple formula

  (18)

(see the Appendix at the end of this document for a proof). With the above results, the general solution becomes

  (19)

Because of the even/odd properties of the trigonometric functions,  and , so the above may be simplified as

  (20)

which is a linear combination of sine and cosine functions. (Compare with Case 1, in which the “interesting result” was a solution consisting of a linear combination of sinh and cosh functions.) If a real (rather than complex) result is required, as is generally the case in physics and engineering problems, we set  and  to be the complex conjugates of each other (that is,  and , with  real). With this restriction, the most general *real* result becomes

  (21)

  (22)

Setting  and , the most general *real* result becomes

  (23)

In the applications, it will be seen that this type of solution predicts the sinusoidal motion of an undamped harmonic oscillator, and the naturally sinusoidal current in an undriven *LC* circuit.

Case 4:  and  (auxiliary equation has two conjugate complex roots). This is the most complicated (and most interesting) case, so we’ve saved it for last. Consider the example

  (24)

The auxiliary equation is , and we may verify

  (25)

The left side of the auxiliary equation is prime (not factorable), obliging us to use the quadratic formula to obtain its solutions. We find

  (26)

which is fortunately not as bad as we might have feared, given the large coefficients in the original problem (perhaps you can tell that I “cooked” the numbers for this example).

The corresponding independent solutions to the ODE are

  and . (27)

From our analysis of Case 3, we know that , and that, if we wish to restrict ourselves to general solutions that are real, we should select parameter values that are complex conjugates of each other. (Note that the factors of  are already real.) So we select  and , and write the general real solution as

  (28)

  (29)

  (30)

  (31)

  (32)

Setting  and  allows the general form of real solutions to be written as

  (33)

The graph of one member of this family of solutions (with  and ) is shown below.



As you can see, these solutions may model the decay of an undriven harmonic oscillator with damping, or of a discharging *LRC* circuit. In either case, the solutions asymptotically approach zero at either the left or right end of the *x*-axis.

Higher-order equations. For equations of order 3 or higher, the auxiliary equation will have the same degree as the order of the corresponding homogeneous linear ODE, and so must be solved by factoring or by approximation methods. (Most graphing calculators will provide approximate real decimal roots of polynomial equations; a few will also provide complex roots.) The rules for determining independent solutions are like those found Cases 1 through 4 derived above. In summary, if we seek only real solutions, then:

* For every real root *m* of the auxiliary equation having multiplicity 1, there will be an independent exponential solution to the homogeneous linear ODE of the form .
* For every real root *m* of the auxiliary equation having multiplicity , there will be *k* independent solutions to the homogeneous linear ODE of the form , , , , …, and .
* If the coefficients of the homogeneous linear ODE are real, every pure imaginary root  will be accompanied by a conjugate root . If these roots each have multiplicity 1, there will be two independent real solutions to the homogeneous linear ODE of the form  and . If these roots each have multiplicity , there will be  independent real solutions of the form , , , …, , , , , …, and .
* If the coefficients of the homogeneous linear ODE are real, every complex root  will be accompanied by a conjugate root . If these roots each have multiplicity 1, there will be two independent real solutions to the homogeneous linear ODE of the form  and . If these roots each have multiplicity , there will be  independent real solutions of the form , , , …, , , , , …, and .

See examples 3 and 4 on page 138 of the textbook.

**Appendix: Proof of Euler’s Formula**

Because the expression  contains the imaginary number *i*, assume that the result is a complex number  for which the values of  and  are as yet unknown; that is,

  (34)

However, because the values of  and  both depend on the value of *x*, we may rewrite this, showing the explicit functional dependence, as

  (35)

Taking the derivative with respect to *x* on both sides of equation (35) (being careful to employ the chain rule correctly as we would for implicit differentiation, and treating *i* as a constant) gives

  (36)

Segregating the terms containing  from the terms containing  and factoring these out gives

  (37)

Changing the negative sign in the second set of braces to  and factoring out one power of *i* from each term gives

  (38)

Comparing the braced expressions with equation (35) and substituting into the above equation gives

  (39)

from which it follows (by directly comparing the two sides of the equation in a manner like that used in the method of partial fractions) that  and  for all *x*. Integrating these results suggests that  and , where  and  are constants. Plugging these back into equation (35) gives

  (40)

We may find numerical values for  and  by noting that if , then . Plugging this into the preceding equation gives

  (41)

Equating the real and imaginary parts of the last equation above gives the system

  (42)

We cannot solve the first equation in this system if ; therefore we assume , which makes it safe to divide the second equation on both sides by , giving  and therefore . Plugging this into the first equation of the system gives  and therefore . Finally, inserting these results into equation (40) gives

  (43)

which is the Euler formula. *Q.E.D*.[[1]](#footnote-1)

The result may also be proved by expanding the Taylor series for , , and  out to at least the 8th-power terms, then replacing *x* with *ix* in the expansion of , collecting real and imaginary terms, and comparing with the  and  expansions, respectively. However, that approach assumes *a priori* that Taylor series are still valid for complex arguments. (It turns out that they are, but the proof requires upper-division mathematics, and would have to be taken on faith for now, whereas the above proof only assumes that when an expression containing *i* is differentiated, we may treat *i* the same way we would treat any real constant.)

1. *Q.E.D.* stands for “quod erat demonstrandum,” a Latin phrase meaning “that which was to be demonstrated.” In older mathematics texts, this abbreviation was frequently inserted f+ollowing each proof as an indication that the proof was complete at that point. [↑](#footnote-ref-1)