MATH V23 LECTURE NOTES (Bowen)  
Section 4.2  
Reduction of Order

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

Reduction of order is a technique used to find one of the two independent solutions of a second-order homogeneous linear ODE when the other independent solution is already known on a certain interval *I* of the real number line (*x*-axis). This will be used in the next section of the textbook [Zill] to handle certain cases when one solution is easy to find, but the other is not. The technique involves multiplying the known solution by a complicated expression that looks a little bit like an integrating factor. In accordance with the theory of linear homogeneous equations developed in section 4.1, the general solution to the second-order equation consists of all linear combinations of the two solutions thus obtained.

Development of the technique of reduction of order. Consider the most general linear ODE of order 2, which is

 (1)

Let us rewrite this as

 (2)

and suppose that  is one of the two independent solutions, and is known. From the theory in textbook section 4.1, we know there is a second (but, for the moment, undetermined) independent solution , which we seek to obtain. From the definition of independence, we know that, for constants  and , the only solution of

 (3)

on the interval *I* is  and . If we solve the above equation for the unknown solution , we obtain

 (4)

from which we may safely conclude that  is *not* a constant multiple of , because there were no nonzero solutions to equation (3). We further conclude that there is some *non-constant* function  such that

 (5)

Because  is, by hypothesis, a solution to the ODE, we should be able to take its derivatives (by applying the product rule to the right side of the above equation) and substitute them into equation (2) to obtain an identity. Let’s take the derivatives first:

 (6)

 (7)

Next, substituting these into equation (2) gives

 (8)

The underlined (red) terms in the above equation add to zero, because  is a known solution of the homogeneous equation, and the sum of the underlined (red) terms (after  is factored out) is simply what we would get on the left side if we substituted that function into equation (2). The remaining terms are therefore

 (9)

which may be separated into terms containing either  or :

 (10)

With the substitution  (and, therefore, ), we may write this as

 (11)

This equation is not only linear and first-order in *w*, but it is also separable (if we move the term containing  to the right side and then divide everything by ):

 (12)

Integrating both sides of this equation (by applying logarithmic differentiation in reverse) yields

 (13)

and consolidating the logarithms gives

 (14)

Exponentiating both sides gives

 (15)

where we justify removing the absolute values by noting that sign of *C* can be adjusted as needed to ensure that . Solving this for  and replacing  with  gives us

 (16)

We may then solve for  by applying a second integration to both sides of this equation, to give

 (17)

We finally obtain the unknown independent solution  by substituting this expression for  into equation (5), and setting  and  to obtain the simplest possible solution, which is

 (18)

In practical problem-solving, we may choose either to employ the above formula directly, or to re-derive the steps leading from equation (5) to equation (18), but using the expressions for  and  that apply to a given problem. Either of the integrals in equation (18) may prove to be non-elementary, in which case we do not attempt to evaluate them, but simply state them as part of the final answer.

Although the formula of equation (18) was derived for homogeneous linear equations of degree 2, it may also be used to help find the general solution for nonhomogeneous linear equations of degree 2. The theory of section 4.1 guarantees that the general solution of the nonhomogeneous linear equation of degree 2 consists of the sum of a linear combination of the independent solutions  and  of the corresponding homogeneous equation, and any particular solution of the nonhomogeneous equation. So, this formula can help us find  if we can determine an expression for .

An example of this method is homework problem #2 in this section, in which we are given the second-order homogeneous equation and one independent solution

 (19)

Comparing this with equation (2), we see that  and . If we chose to find  by re-creating the steps from equation (5) to equation (18), we would start by assuming that the second independent solution could be written in the form  for some non-constant function . To plug this into equation, we need to find the derivatives of . For a triple product of the form , the first derivative is

 (20)

where the explicit functional dependence on *x* has been omitted for conciseness and clarity. For the specific case given above, we find

 (21)

and

 (22)

Direct substitution of  and its derivatives into the original problem gives

 (23)

Distributing the 2 in front of the second set of brackets, factoring out (and then dividing by) , and combining like terms gives

 (24)

 (25)

 (26)

In the reduction-of-order step, we temporarily substitute  and  to obtain

 (27)

which is separable. We subtract 2*w* from both sides of the equation, and multiply both sides by  to obtain

 (28)

A subsequent integration and exponentiation gives

 (29)

for which we justify removal of the absolute values (1) by converting  to , then absorbing the plus-or-minus sign into ; and (2) by noting that  is inherently nonnegative.

To complete the solution, we must convert *w* back to , and then use this to find . Writing the previous equation in terms of  and integrating again gives

 (30)

where the negative sign from the integration is again absorbed into . The solution is

 (31)

If this had been an odd-numbered problem, the textbook authors would have made the following additional adjustments to the answer found in the back of the textbook:

1. Since the second term  is just a multiple of the given solution , it adds nothing new to the second solution (relative to what was already given to us), so it would be omitted due to its not being a novel solution of the homogeneous equation.
2. Because it is understood that the general solution will be a linear combination of the form , the constant of integration, or parameter,  in front of the first term would also be omitted, leaving us with simply  as the textbook’s final answer.