MATH V23 LECTURE NOTES (Bowen)  
Section 4.1  
Preliminary Theory—Linear Equations

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

In Chapter 2 of the textbook [Zill], we developed techniques to solve first-order linear ODEs. In Chapter 4, we will develop additional techniques for characterizing and solving linear ODEs of second or higher orders. We will be seeking general solutions, which include parameterized solution families as well as singular solutions. In addition, we will be seeking solutions to initial-value problems (IVPs, as defined in section 1.2 of the textbook) and a new class of problems known as ***boundary-value problems*** (BVPs, to be defined shortly).

The *n*th-order initial-value problem. The most general linear ODE of order *n* may be written in the form

 (1)

If , that is, if

 (2)

then this is called a ***homogeneous equation***; otherwise, it is a ***nonhomogeneous equation***. Regardless of whether an equation is homogeneous or nonhomogeneous, we will find that a necessary part of the solution will be to analyze the homogeneous case; otherwise any solution we obtain will not be fully general.

We may convert the linear equation above into an IVP by prescribing additional conditions for the values of the solution and its first  derivatives evaluated at a *single* point :

 (3)

where  are all constants. Don’t forget that the interval of validity *I* must contain the ordered pair , and the solution curve must have a slope of  at this point.

Theorem 4.1.1 in the textbook guarantees that a solution  exists, and is unique, on some interval containing , provided that  are continuous on this interval, and provided that the first “coefficient”  for every *x* on the interval. To simplify the remaining discussion for this section, we will assume that the conditions necessary to guarantee existence and uniqueness of a solution for an IVP, as described in Theorem 4.1.1, prevail for each equation we analyze.

The *n*th-order boundary-value problem. If the restrictions in equation (3) were changed so that it specified the value of  and/or its derivatives at several (*n*) *different* values of *x*, then it would be recategorized as a ***boundary value problem*** (BVP). The restrictions are known as ***boundary conditions***. Unlike an IVP, which has exactly one solution (if it meets the conditions in Theorem 4.1.1), a BVP may have zero, one, or many distinct solutions. Example 3 in the textbook illustrates this idea.

Differential operators. Because the derivatives in equation (1) can be cumbersome to write, we sometimes use the ***differential operator*** *D* to write them more compactly. So  becomes *,* becomes , and so on. Suppose that *a* and *b* are arbitrary constants, and  and are arbitrary differentiable functions. Applying the differential operator to the linear combination  gives

 (4)

Not every operator has the property that

 (5)

but any operator that does is called a ***linear operator***. An example of a nonlinear operator would be the operator *S* that converts a function into its own square; the reason is that

 (6)

whereas

 (7)

Since the right sides of equations (6) and (7) are unequal, *S* is a nonlinear operator.

Using this notation, equation (1) may be rewritten

 (8)

By “factoring out” the *y* from the above equation, we may also write

 (9)

This gives rise to the concept that the expression in brackets above is a “fancier” operator known as an ***nth-order differential operator***. It can readily be shown that

 (10)

is also a linear operator; hence the designation *L* (for “linear”). So, the linear ODE from equation (1) can be formally abbreviated as

 (11)

or, if the linear ODE is homogeneous, as

 (12)

In the context of differential equations, it is customary to represent functions (which “operate” on numbers) as lowercase letters such as *f* or *g*, and operators (which “operate” on functions) as uppercase letters such as *L* or *T*.

Linear combinations. If we are given a nonempty set of mathematical items such as vectors or functions, then a ***linear combination*** of the elements of the set is defined as an expression constructed from the set by multiplying each term by an arbitrary constant, and adding the multiplied terms. So, a linear combination of the set of functions  would be any expression of the form

 (13)

Superposition principle for homogeneous equations. The linearity of *L* from equation (10) leads to a useful result. For any *homogeneous* linear ODE , if each of  is a known solution on some interval *I*, then the linear combination

 (14)

where  are arbitrary constants, is also a solution. The textbook provides a limited proof in Theorem 4.1.2, but a general proof is only slightly more difficult. Consider that

 (15)

Because we were told that each of  is a solution, it follows that

 (16)

and plugging these into equation (15) gives

 (17)

thereby proving that  is also a solution of . If we set all the constants equal to zero, it immediately follows that  (the ***trivial solution***) is a solution to every homogeneous linear ODE. If, however, we set all the constants equal to zero (except for *one* of them, which is set to an arbitrary nonzero value), we see that any constant multiple of a solution is also a solution.

Linear dependence and linear independence. A set of functions  is said to be ***linearly dependent*** if a set of constants , *not* all zero, can be found satisfying

 (18)

for *every* *x* on some interval *I*. If, however, the *only* solution to equation (18) that is valid for every *x* on *I* is , then the set of functions  is said to be ***linearly independent***. A given set of functions must be either linearly dependent or linearly independent; there is no third category of “other” sets. Note that any set of functions containing the element  is necessarily linearly dependent. If the set contains only two functions, then to demonstrate linear independence, it is sufficient to show that the functions are not constant multiples of each other. For a larger set of functions, the set is linearly independent if no one function in the set can be expressed as a linear combination of the other functions in the set.

As it can be difficult to determine whether a given set of functions (especially a large set) is linearly independent, there is fortunately a mechanical means of doing so, if each of the *n* functions in the set is differentiable at least  times on some interval *I*. Consider the determinant

 (19)

which is called the ***Wronskian*** of the set of functions. Theorem 4.1.3 in the textbook guarantees that if (and only if) the Wronskian is nonzero for every *x* on , then the set of functions is linearly independent on . From the if-and-only-if relationship, we create the following chain of logic:

* If, for any *x* on *I*, the value of *W* is zero, then the set of functions is linearly dependent.
* If the set of functions is linearly dependent, then the value of *W* for any other *x* on *I* must also be zero.
* So, if we compute *W* for any single value of *x* on *I*, and discover that the result is nonzero, then the set of functions cannot be linearly dependent, therefore the set is linearly independent.
* If the set is linearly independent, then the value of *W* for any other value of *x* on *I* must also be nonzero; otherwise we would conclude (from the first bullet in this chain) that the set was linearly dependent.

Therefore, either  (the set is linearly dependent) for every *x* on *I*, or  (the set is linearly independent) for every *x* on *I*. This is helpful from a computational standpoint, because to prove either linear dependence or linear independence, it is not necessary to compute *W* for every *x* on *I*. We may instead select a single value of *x* from *I*, and find *W* for that single value. If  for just that one value of *x*, then the set of functions is linearly dependent on the entire interval *I* that contains *x*, and if  for just that one value of *x*, then the set of functions is linearly independent on the entire interval *I*. This simplification is illustrated in the textbook in the unnumbered example on the top half of page 125, where the behavior of *W* at a single point  is used to prove that a set of solutions to a given homogeneous equation is independent on the interval  that contains .

One requirement of obtaining a minimal yet comprehensive family of *n* solutions (also known as a ***fundamental set*** of solutions) for a homogeneous linear ODE of order *n* is that the solutions must compose a linearly independent set. So, the idea of linear independence will become of great importance in later sections, when we begin to find solutions to higher-order linear ODEs. Incidentally, there is a connection here between polynomials of degree *n* (which are guaranteed by the Fundamental Theorem of Algebra to have *n* complex solutions) and homogeneous linear ODEs of order *n* (which are guaranteed to have *n* linearly independent solutions).

Theorem 4.1.4 in the textbook guarantees that a fundamental set of solutions exists, at least on an interval *I*, for every homogeneous linear ODE. Theorem 4.1.5 combines Theorems 4.1.2 and 4.1.4 to specify the general solution, which is

 (20)

where  is a fundamental set of solutions, and  is a set of arbitrary constants. Conversely, any solution  that is already known to exist can be rewritten as a linear combination  of the elements of a set of fundamental solutions.

Note (for those students who have already taken linear algebra): The set of all solutions of a homogeneous linear equation constitutes an abstract vector space of dimension *n*. A set of *n* linearly independent solutions forms a basis set for that vector space. So, every solution is a linear combination of the basis functions, and every linear combination of the basis functions is a member of the solution set (that is, a vector in the vector space).

Nonhomogeneous equations. Any function  that satisfies the nonhomogeneous form in equation (1) is called a ***particular solution*** of the equation. The general solution is a linear combination of the fundamental set, plus the particular solutions (Theorem 4.1.6).

A different kind of superposition. Suppose that the right side of a nonhomogeneous linear ODE is a sum of several functions. For example, suppose that

 (21)

It may be difficult to find a particular solution for this equation. However, if we can break the problem into pieces by finding particular solutions , , and  satisfying

, , and  (22)

then Theorem 4.1.7 guarantees (by employing the linearity of *L*) that  is a particular solution for equation (21).