MATH V23 LECTURE NOTES (Bowen)
Section 3.1
Linear Models

Now that we have learned to solve several types of first-order ODEs, we may revisit several of the models developed in section 1.3; this time we won’t just be setting up the equations, but solving them.

Growth and decay.



This is used for population growth and radioactive dating, where *t* (time) is the independent variable, *P* (population or number of atoms or number of grams/kilograms/moles) is the dependent variable, and *k* is a positive constant (for growth) or a negative constant (for decay). For radioactive decay, *k* is related to the half-life through the relationship .

Solution method: Separable equation.

Newton’s law of cooling.



This is used for warming and cooling problems, where *t* (time) is the independent variable, *T* (temperature of the cooling or warming object, in kelvins/degrees C/degrees F) is the dependent variable, *Tm* is the temperature of the environment (may be constant or a function of time, measured in the same units as *T*), and *k* is a positive constant (for a cold object warming up) or a negative constant (for a hot object cooling down).

Solution method: Separable equation.

*LR*-series circuit.



This is used for a single-loop circuit containing an inductor *L* (in henrys), resistor *R* (in ohms), and power supply EMF (not electric field), or input  (in volts); and for which *t* (time, usually in seconds) is the independent variable, and electric current *i* (usually in amperes) is the dependent variable. For a DC circuit (the simplest to solve),  is a constant for all values of *t*. A variation on the DC circuit is the step function



where  is a constant. which can (after being inserted into the differential equation above) model the behavior of a DC circuit as it is turned from the “off” state to the “on” state at time . Circuits containing nonlinear elements such as inductors or capacitors often exhibit short-term behavior (the “transient response”) which differs significantly from the long-term behavior (the “asymptotic response” or the “steady-state response”). The transient response typically manifests itself when the input state  experiences a sudden change, such as turning the power off or on, and then quickly approaches zero after a fraction of a second, leaving only the asymptotic response to dominate the response function for large values of *t*.

If the input is sinusoidal (AC source), then the asymptotic response is also sinusoidal, oscillating at the same frequency as the input, but is generally not in phase with the input (the input EMF reaches its cyclic peaks a fraction of a period before the response current; this phase difference is greatest at high frequencies), as can be determined by solving the differential equation for . The amplitude of the response depends on the interaction between *L*, *R*, and the input frequency; for fixed values of *L*, *R*, and the input amplitude, solving the differential equation demonstrates that the response (current) is lower at high frequencies, and higher at low frequencies. For this reason, *LR*-series circuits are sometimes called “low-pass filters.” In the limit as frequency approaches zero, the inductor has minimal effect on the circuit behavior, and the circuit may be modeled as though only a resistor were present. The transient response, although short-lived, can briefly produce currents many times higher than the steady-state response; in an unprotected circuit, the transients can shorten the lives of circuit components, and even burn them out in a single incident.

In many cases, *L* and *R* may be considered constant; however, there is at least one homework problem in which *L* is treated as a varying function of time. It is perhaps not surprising that time-varying values of *L* or *R* typically complicate the solution of the model differential equation.

Solution method: Integrating factor (DC circuits); more advanced methods not yet covered (AC circuits).

*RC*-series circuit.



This is used for a single-loop circuit containing a capacitor *C* (in farads), resistor *R* (in ohms), and power supply EMF, or input  (in volts); and for which *t* (time, usually in seconds) is the independent variable, and electric charge *q* (usually in coulombs) is the dependent variable, although in practical situations, the current  may be of greater interest (it is obtained by solving the above equation first for *q*, then differentiating with respect to *t* to get the current *i*). For a DC circuit (the simplest to solve),  is a constant for all values of *t*. The *RC*-series circuit has both transient and asymptotic responses, as described previously for the *LR*-series circuit.

For an AC circuit, assuming fixed values of *C*, *R*, and the input amplitude, solving the differential equation demonstrates that the asymptotic response is also sinusoidal, oscillating at the same frequency as the input, but is generally not in phase with the input (the input EMF reaches its cyclic peaks a fraction of a period after the response current; this phase difference is greatest at low frequencies). Also, the amplitude of the response (current) is higher at high frequencies, and lower at low frequencies. For this reason, *RC*-series circuits are sometimes called “high-pass filters.” In the limit as frequency approaches infinity, the capacitor has minimal effect on the circuit behavior, and the circuit may be modeled as though only a resistor were present.

Solution method: Integrating factor (DC circuits); more advanced methods not yet covered (AC circuits).