MATH V23 LECTURE NOTES (Bowen)
Section 2.4
Exact Equations

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

In the previous section, we examined first-order linear ODEs. In this section, we develop another technique that also works for some first-order linear ODEs.

Generating an exact equation from its solution. Consider the broad class of equations that may be written in the form

  (1)

where *c* is a constant parameter. An example of such an equation might be

  (2)

where  and . For each fixed value of *c* for which the equation has at least one solution , the graph of this curve is (in the language of MATH V21C) a ***level curve*** of the function . If you are not familiar with this terminology, an alternative is to think of the *xy*-plane as a topographic map laid on a tabletop, with  representing the altitude of point  above sea level. On such a map, the graph of an equation such as  could represent the contour line passing through every map point having an elevation of 4 meters above sea level. The graph of the related equation  could likewise represent the contour line passing through all points located 8 meters above sea level, and so on.

If it were possible to walk along a contour line on the Earth’s surface, then, because every point we visited would be at the same elevation relative to sea level, we would never move uphill or downhill at any point during the walk, relative to our starting elevation. With no change in the value of *z* or  occurring along the contour, the exact differential (“rise”; see the [Review of Multivariable Calculus Concepts](http://academic.venturacollege.edu/mbowen/courses/handouts/h_v23_multivariable_review.docx)) of  is zero, leading immediately to equation (1) in the textbook [Zill]):

  (3)

So, as shown below, we may generate a first-order differential equation by calculating the exact differential on both sides of equation (2), using the formula in equation (3):

  (4)

The last result in equation (4) (following the arrow) is said to be an ***exact equation***, because the left side is an exact differential. This equation was obtained by taking the exact differential of , so we already have the family of solutions; it’s simply . In other words, the solution of equation (4) is just equation (2)! This may seem backwards (we started with the solution and generated the resulting first-order differential equation), but it’s a step in the right direction; if we could just run this process in reverse, then we could readily solve any equation that was known to be exact.

Identifying and solving exact equations. From the result of equation (3), we see that an exact equation may also be written in the form

  (5)

where  and . However, not every expression in the form of equation (5) is exact, since  and  may be chosen arbitrarily, and are therefore not guaranteed *a priori* to arise from the computation of an exact differential. So how do we know whether an equation such as this is exact? Let us examine the partial derivatives  and :

  (6)

These turn out to be the mixed partials of , which a theorem guarantees to be equal, provided they are defined and arise from the same function . So, we may identify exact equations by computing  and ; if (and only if) they are equal, then equation (5) is exact (see also equation (4) in the textbook). An exact equation may be solved by integrating  with respect to *x* and  with respect to *y*, then reconciling the “constants” (functions, actually) of integration to obtain the implicit solution . The value of *c* may then be adjusted to satisfy initial conditions in the case of an IVP. Because we already know that the solution of equation (4) is equation (2), integrating the former to obtain the latter would likely not create much additional understanding for the student. So, a different example illustrating these integration processes is provided for another exact-equation problem, beginning with equation (15) below.

Extending this method to certain types of non-exact equations. A non-exact equation can be made exact, in some instances, by borrowing from the method of linear equations in section 2.3, and multiplying both sides of equation (5) by an integrating factor . To determine whether an integrating factor can be used to convert a non-exact equation  into an exact equation, compute the value of the expressions

  and  (7)

If the first expression in equation (7) is a function of *x* alone (contains no occurrences of *y*), then use the integrating factor

  (8)

to convert the non-exact equation to an exact equation.

If the second expression in equation (7) is a function of y alone (contains no occurrences of *x*), then use the integrating factor

  (9)

to convert the non-exact equation to an exact equation.

Complete solution of a non-exact equation (includes solution to the exact equation resulting from multiplication by an integrating factor). Consider the differential equation

  (10)

This equation is *non-exact*, because

  and  (11)

so . We next compute  to determine whether there is an integrating factor that is a function of *x* only:

  (12)

The last result is not a function of *x*, so we then try to compute  to determine whether there is an integrating factor that is a function of *y* only.

  (13)

This is a function of *y*! Note that the result of equation (13) already provides us with the value of a significant portion of the integrating factor  whose formula comes from equation (9). Substituting equation (13) into equation (9) gives

  (14)

Multiplying the original problem by the integrating factor  gives

  (15)

which is now exact (verify this yourself by computing and comparing  and ; use the updated functions *M* and *N* from equation (15), not the original ones from equation (10)).

The solution of equation (15), below, shows the steps to follow if the original equation you are given is exact. The steps shown from equations (7) through (14) are only necessary if the original equation is non-exact.

The next step is to integrate to find . Because the “constants” of integration generalize to become functions of the constant variable from each integration, we must integrate with respect to both *x* and *y*, and then use the functions of a constant variable to reconcile the two solutions into a single function. We begin by integrating with respect to *x* to obtain

  (16)

where is the “constant” of integration (any expression containing only *y* is a constant when *x* is the independent variable). We follow up by also integrating with respect to *y* to obtain

  (17)

Because the left side of equation (15) is an exact differential of , the results for  in both equations (16) and (17) must agree with each other (we must reconcile them into a single expression). In comparing the two results, we note that both contain a term  that is a function of both *x* and *y*, so this must be one of the terms in the consolidated solution. In equation (17), there is a term  that does not appear explicitly in equation (16); however, it is a function of *y* only, so it corresponds to the term  in equation (16). There is no term in equation (16) corresponding to the term  in equation (17), so we set  and ignore it in the reconciliation. (We don’t insert a constant *C* for a reason that will become apparent shortly.) With the above substitutions, we find that

  (18)

We close the loop by observing (from equation (1)) the correct form in which to write the family of solutions; we obtain

  (19)

as the general solution to both equation (10) and equation (15). If we had also been given an initial condition (making the problem into an IVP), such as , we could plug in the values of *x* and *y* to determine a specific value of the parameter *C*:

  (20)

which would lead directly to the specific solution

  (21)