MATH V23 LECTURE NOTES (Bowen)  
Section 2.3  
Linear Equations

Unless otherwise stated, it will be assumed in this section that *x* is the independent variable, and *y* is the dependent variable, for each differential equation discussed.

The following discussion contains equation numbers. References to equation numbers point to the equation numbers within these notes, and do *not* correspond to the equation numbering in this section of the textbook, unless an explicit reference to the textbook appears along with the stated equation number.

We will now examine first-order linear ODEs, which the textbook [Zill] abbreviates as “linear equations.” (Linear ODEs of second or higher order were briefly introduced in section 1.1, but their solutions are sufficiently difficult that we will not attempt them here). Some (but not all) first-order linear ODEs are also separable, in which case they should be solved using the method of section 2.2. But what if such an ODE is not separable? It turns out that a general method for solving it exists for a broad class of functions. The method relies upon a clever manipulation that takes advantage of a well-known property of derivatives.

Terminology, definitions, and forms of a first-order linear ODE. A first-order linear ODE is an equation of the form

 (1)

The function  on the right-hand side is called the ***source term***. This terminology derives from the theory of electrical circuits, in which this function would model the electromotive force (EMF, or voltage) of the circuit’s power source. (In that context, the independent variable would be *t*, not *x*, and *g* would describe how the EMF varied over time. The other terms in the equation would typically model the responses of other circuit elements.)

If , then the equation is said to be a ***homogeneous equation***; otherwise, it is a ***nonhomogeneous equation*** (or an ***inhomogeneous equation***). These terminologies parallel what you may have seen in linear algebra, where the ***homogeneous matrix equation***  is an important special case. The solutions to a homogeneous equation are called ***complementary functions***, and, as in linear algebra, we will see later that they are useful in constructing a complete set of general solutions for the corresponding nonhomogeneous equation.

The expressions  and  are also functions of *x*, and may be chosen arbitrarily (including both linear and nonlinear functions of *x*), subject only to the restriction that both must be differentiable. If both  and  are constant functions, the resulting linear ODE is said to have ***constant coefficients***.

Dividing both sides of the first-order linear ODE by the ***lead coefficient***  gives rise to another form of this equation called the ***standard form***, an alternative which is often more convenient for obtaining solutions. The standard form looks like

 (2)

and the interval of validity for solutions *y* may only include regions of the *x*-axis for which the ***coefficient functions***  and  are both continuous. Adapting terminology from engineering fields, the function  is sometimes called the ***input function*** or the ***driving function***; a solution y is sometimes called the ***output*** or ***response***.

Finding the integrating factor. Starting with the standard form of the equation, it is possible to transform the left side of the equation (through multiplying both sides by a suitable function , called an ***integrating factor***) into an expression having the same structure as the product rule () for derivatives. Because the product rule is a “perfect derivative” (namely, the derivative of the product ()), it can be integrated (and thereby a solution found) with relative ease. The symbol  (“mu”) is a Greek lowercase “m,” and likely chosen for this purpose by the textbook author because “m” is also the first letter of “multiplication.” The same symbol is used in the International System of measurements (SI) as the abbreviation for “micro-” ().

Typically, the most difficult part of solving the first-order linear ODE is finding the correct integrating factor to use. The logic we use to derive the correct  has some twists and turns in it; it might help to understand that logic by reviewing the method of partial fractions from MATH V21B. As you may recall, this reverses the technique of adding rational expressions via an LCD, splitting a complicated fraction into two or more simpler fractions, in hopes that the latter will be easier to integrate than the original. For example, if we needed to find an antiderivative of

 (3)

we would first set up a partial fractions equation by setting the right side of equation (3) equal to the skeletons of some simpler fractions:

 (4)

The left side of equation (4) bears only a slight resemblance to the right side; moreover, if we used the “guess and check” method, repeatedly picking random values to substitute in for *A* and *B*, the two sides of this equation would almost *never* be equal (which makes sense, in a way; how could two things be equal if they look different from each other?).

But that is the magic of algebra: it lets us place two different-looking expressions side-by-side, assert that, even though they are unequal most of the time, there might be rare circumstances under which they *are* nevertheless equal, and then discover what those specific circumstances are, by solving for the specific values of *A* and *B* that “work.” In a sense, determining the unique solutions for *A* and *B* amounts to investigating the consequences of our seemingly rash decision to equate two dissimilar quantities.

Returning to the determination of , we start by considering the derivative of the product , where  is the correct integrating factor (unknown at this stage), and *y* is one of the solutions being sought (also unknown). This derivative is

 (5)

Because the left side, being the derivative of the quantity in brackets, is easy to integrate, the right side, being the same quantity (just written differently), should also be easy to integrate. Now, if we could just manipulate the left side of equation (2) to look like the right side of equation (5), we’d be able to integrate that as well, which would, in turn, enable us to find the solution *y*. To begin this manipulation, we multiply the left side of equation (2) by :

 (6)

We note that the first terms in equations (5) and (6) (the terms shown in blue) are the same, but the second terms (the terms shown in red) appear not to be. This is initially discouraging, but, just as in the partial fractions discussion, we may use algebra to place an equals sign between the two quantities (the red terms in equations (5) and (6)) that we would like to be equal, solve for any  functions that make the asserted equality true, and then finish up by integrating equation (6) to find *y*. Equating the red terms in equations (5) and (6) gives

 (7)

from which it appears we may divide both sides by *y* to obtain

 (8)

which (surprise!) is separable; multiply both sides by  to obtain

 (9)

which is ready to integrate:

 (10)

To isolate the integrating factor , we move the constant of integration to the right side of the equation, remove the natural logarithm by exponentiating both sides of the equation, and remove the absolute value symbol by introducing a  symbol:

 (11)

where . Because any value of  yields a suitable integrating factor , we arbitrarily set  (for simplicity); the final answer is

 (12)

Because it can be inconvenient to write  as a small exponent, we define a new three-letter abbreviation (“TLA”): the exponential function . With this notation, equation (12) may be rewritten more conveniently as

 (13)

Equations (12) and (13) are the central equations of this section; one or both should be prominently visible on the “cheat sheet” you prepare for the first examination.

General method of solution of a first-order linear ODE: derivation. Even with all this work, we haven’t solved the original first-order linear ODE. This subsection of the notes is a discussion of the theory. It’s analogous to *deriving* the quadratic formula for the first time; *using* the formula in practice is a different process, which we will get to later in this section.

To come up with a formula for the solution, we plug either of equations (12) or (13) into equation (6) to obtain

 (14)

Equation (5) claims that the left side of equation (14) is equal to ; let’s verify that claim:

 (15)

The chain rule requires, in general, that , so we simplify the last term in equation (15) as

 (16)

The Fundamental Theorem of Calculus allows us to simplify the last factor in this equation to simply ; substituting this and rearranging the factors in accordance with the commutative property for multiplication gives

 (17)

The right side of equation (17) matches the left side of equation (14), thus verifying the claim of equation (5). Plugging equation (17) into the left side of equation (14) allows us to write the latter as

 (18)

We’re almost there! (In case you lost track of where we’re heading, we are trying to isolate *y*.) Integrate both sides of the preceding equation, and apply the Fundamental Theorem of Calculus to simplify the left side:

 (19)

 (20)

where we’ve moved the constant of integration to the right side. Finally, multiply both sides by  to obtain a family of solutions for *y*:

 (21)

Normally we’d put a box around this, but this result is too horrendous to memorize or write on a cheat sheet, so I decided not to. In practice, we’ll use equations (12) and (13) to solve our first-order linear ODEs, but the above derivation provides a pathway to follow to the solution (and assures us that a solution exists). This corresponds to equation (4) in the textbook.

General method of solution of a first-order linear ODE: application. The steps for solving equation (2) are summarized in the textbook, in a box above Example 1 on page 56. This method handles both general solutions and initial-value problems, as illustrated in the numerous textbook examples.

As an additional example, let’s solve an equation that models the velocity of an object falling under the influence of gravity, but also considers the effects of air resistance. Such problems were likely discussed, but probably not explicitly solved, in your PHYS V04 class. From your experience there, you might understand that a falling object tends to accelerate until it reaches a stable speed of fall, called the ***terminal velocity***. This applies both to objects that are dropped and to objects that enter the atmosphere at a high rate of speed, such as space shuttles and asteroids (which must slow down, rather than speed up, to achieve their terminal velocity). If we ignore the effects of altitude on the strength of gravity (that is, we start with a low-resolution model), we may represent that as , where we are assuming units of meters and seconds (hence the familiar value of 9.8), and temporarily ignoring air resistance;  is a symbolic way to represent the concept “acceleration of gravity.”

To model the air resistance, we consider the familiar experience of opening the window of an automobile while riding inside, and extending our arm or hand outside the window. The faster the car travels, the greater the force of air resistance against our hand and arm. For a falling body, that force would produce an acceleration in the direction opposite the velocity, suggesting that the relationship between velocity and acceleration should contain a negative sign. To keep the model (and the resulting ODE) linear, we assume that the force of air resistance is directly proportional to the velocity, leading to a model of the form , where *b* is a positive, constant “fudge factor” whose value depends on the mass, size, and shape of the object, and  is the component of the overall acceleration attributable to air resistance. (A more realistic high-resolution model would look like , and account for the possibility that *b* might change as the object rotated in flight or, in the case of an asteroid, began to vaporize.) For the sake of argument, let’s suppose that the value of this fudge factor for the falling object we’ve chosen is . Considering both gravity and air resistance, with the combined (added) accelerations from both effects defined (in the usual way) as , and adopting the convention that a downward velocity is negative, our model equation becomes

 (22)

and we can see that, for this example, *v* is the dependent variable, *t* is the independent variable, and our long-term goal is therefore to solve for *v*.

To make this model resemble the form of equation (2) (that is, standard form), with the understanding (from the preceding sentence) that *v* is like *y*, and *t* is like *x*, we add  to both sides to give

 (23)

which (in case you didn’t notice) is the special case of an equation with constant coefficients. We therefore have  and , and the integrating factor (from equation (13)) becomes

 (24)

Note that, per the discussion leading up to equation (12), it is safe to ignore (in the context of computing the integrating factor) any constants of integration which may arise, which is why the preceding equation does not have a *C* in it. We multiply both sides of equation (23) by the integrating factor found above to obtain

 (25)

The left side of this equation is, by construction, equal to the derivative , although you should verify this yourself. (Getting the equation into a form such that the left side is the derivative of  or, more generally, of , is the reason why we invented the integrating factor in the first place.) Rewriting the above using the front end of the product rule, we obtain

 (26)

which we immediately integrate with respect to *t* to obtain

 (27)

The general family of solutions is obtained by multiplying both sides of the last equation by :

 (28)

Note the important role that is played by the constant of integration in the result. We could turn this into an initial-value problem by specifying the velocity at a single moment in time. For example, if the object were simply dropped at time zero (that is, if it were initially at rest), then we could state  as the initial condition. Plugging this into the general solution would give

 (29)

The specific solution, obtained by plugging this value of *C* into the general solution, would be

 (30)

Note that for *every* member of the general family of solutions, the velocity for large values of *t* approaches . This represents the terminal velocity, after the object has been falling for a long period of time. You should verify this by using wxMaxima’s “plotdf” function (or the analogous function in Maple or Mathematica, if you have these) to create a direction field and representative solutions for this ODE. Because “plotdf” assumes that the left side of the ODE is  or , you should use equation (22), not equation (23), and replace *v* with *y*.

Please read the textbook regarding solutions for piecewise input functions. These are not merely a theoretical curiosity; these functions can model an electric circuit with a DC power supply that is turned on (1) or off (0) at time zero.

Textbook errata:

Figure 2.3.3 on page 60 should have an open circle on the graph at .