MATH V23 LECTURE NOTES (Bowen)
Section 1.3
Differential Equations as Mathematical Models

Mathematical models. The behavior of real-life systems or phenomena can often be described and/or predicted because observables (*e.g.*, the position or velocity of a projectile, the human population of the Earth, the gross domestic product of a nation, or the number of radioactive decays per second in a sample of plutonium) closely resemble mathematical functions. A description of the behavior in terms of these functions is called a ***mathematical model***.

Resolution. Mathematical models may vary in complexity, from simple or ***low-resolution*** (containing one or just a few variables) to very sophisticated or ***high-resolution*** (possibly containing thousands of variables). Supercomputers may be used to compute the predictions of the most complicated models, such as climate models. Simple models ignore most of a system’s variables, often resulting in somewhat inaccurate representations of a system or phenomenon. For the motion of a projectile, a simple model might only consider the force of gravity (assumed constant). Increasingly sophisticated models may attempt to account for air resistance (including the effects of prevailing winds and/or changes in the projectile’s orientation), variations in elevation of the terrain, variations in the acceleration of gravity due to altitude, the curvature of the Earth’s surface, and/or the rotation of the Earth.

Model development and refinement. Models typically incorporate known or suspected laws governing the behavior of a system. These laws often specify how the rates of change of observables are related to the values of the same or other observables. Because derivatives are used to state rates of change, the resulting model often involves derivatives, and can result in the creation of one or more differential equations to model the system. Solving these equations allows us to compare the model’s predictions with the behavior of the system being modeled. If predictions differ significantly from observations, we may need to increase the resolution of the model, re-examine our assumptions, or both. Figure 1.3.1 in the text [Zill] shows the iterative process through models may be refined. High-resolution models are costly (you need smart people to develop them and/or significant computing power to test them); there is an engineering trade-off between resolution and expense that ultimately limits the accuracy of any practical model.

The textbook [Zill] discusses models of population dynamics, radioactive decay, Newton’s law of cooling, the spread of a disease, chemical reactions, mixtures, draining a tank, series circuits, falling bodies, and the shape of a suspension cable. Each may be modeled by one or more differential equations. In the homework, you will be asked to create differential equations to describe these and other phenomena. You may be asked to consider effects (variables) not initially addressed in the textbook’s discussion. You are not yet expected to solve these equations, but just to write them down, including a brief analysis or justification.

Population dynamics. For population growth, a simple model may only take birth rates into account, while ignoring deaths, migration, or variations in the birth rate over time due to war, disease, famine, aging of the population, or improvements in medical and sanitation practices. Because populations change over time *t*, it is reasonable to select *t* as the independent variable for the model, and the existing population *p* (whose behavior we are studying) as the dependent variable. Most models assume that more babies are born in large cities each year than in small villages because (on average) every existing human produces new humans (babies) at a fixed rate. If, for example, a model assumes that 2 babies per year are born for every 100 people already existing during the year, then a village of 1000 people can reasonably expect about 20 births per year, while a metropolis of ten million people can reasonably expect about 200,000 births in the same period. This type of relationship, in which one variable (*b* = births per year) increases linearly with another variable (*p* = existing population) is called a ***direct proportionality***, and may be written



where the symbol  is read “proportional to,” and *k* is a positive constant (the birth rate). Because the number of births represents a change in the existing population, the number of births *per year* represents the *rate of change* of *p*; written symbolically, . The last equation above becomes (with this substitution)



and a new ODE is born. A higher-resolution model might add other variables and equations to represent death rates, migration rates, and other smaller effects on the population level. Later in the course, we will solve the equations produced by our models, but for now it is sufficient to write the equation, and so this simple model is complete.

Radioactive decay. Radioactivity, although a completely different process from population growth, may be modeled by essentially the same ODE. If Lump A of plutonium has a mass of 10 grams, and Lump B of plutonium has a mass of 80 grams, it seems reasonable to expect that Lump B will exhibit eight times more radioactivity than Lump A. We let time *t* be the independent variable, and let the mass *m* of plutonium in the lump be the dependent variable. The level of radioactivity (number of nuclear decays per second, or *r*) is therefore directly proportional to the mass *m*, and we may write



Since each decay (that is, each “click” of the Geiger counter) represents a plutonium nucleus spontaneously changing into another type of nucleus, the mass of plutonium in each lump is decreasing over time, at a rate  that is equal to the decay rate *r*. Replacing *r* with  above gives



where *k* is a constant related to the decay rate. The only difference between this and the population model is that, since *m* is decreasing with the passage of time,  is necessarily a decreasing function, which means , and therefore *k*, are negative quantities in this context. Higher-resolution models might incorporate contributions to the radioactivity from non-plutonium contaminants (*e.g.*, uranium, thorium, or unstable isotopes of lead) in the sample.

Newton’s law of cooling. This also involves a proportional relationship, but it is slightly more complicated than the previous examples. A hot object is placed in a relatively temperate (*i.e.*, room-temperature) environment, and allowed to cool naturally over time. The hot object cools rather quickly at first, due to the large difference between its temperature and that of the environment, but the rate of change of its temperature gradually decreases as it approaches room temperature. In the simplest models, the room is assumed to be large enough that its temperature does not increase appreciably as it absorbs the thermal energy lost by the hot object; one of the homework problems asks you to develop a higher-resolution model that also adjusts for changes in room temperature.

The temperature of the once-hot object ceases to change once it becomes equal to the temperature of the surroundings. We let time *t* be the independent variable, and the temperature *T* of the hot object be the dependent variable. A third quantity *T*s (the temperature of the surroundings) is also involved. In a low-resolution model, this is assumed constant. Newton’s law posits that the rate of change of the hot object’s temperature is proportional to the difference between the hot object’s temperature and the temperature of the environment, or . Replacing the proportionality symbol with an equals sign and a proportionality constant *k*, as in the preceding examples, gives a model ODE:



The specific value of *k* depends on the shape, size, and composition of the hot object, as well as the ability of the surrounding material to conduct, convect, or radiate thermal energy.

Series circuits. This model describes the flow of electric charge in a single-loop circuit in which all the components (resistor, capacitor, and inductor) are connected in a straight line, like beads on a string. Such circuits are typically studied in the latter half of Physics V05. The rate at which electric charges (typically electrons) pass a given point is called the ***electric current***. The amount of electric charge may be measured using the ***coulomb***, a large unit which corresponds to the amount of charge carried by about  protons (or the same number of electrons, if you disregard that electrons carry a negative charge). If one coulomb of moving charge passes by a given point every second, then the electric current is said to have a value of one ***ampere***. For this reason, we may say that current is the time rate of change of charge, or



(Aside: some people like to say things like, “Turn up the amperage (or amps)!” when they really mean “Turn up the current!” This language confuses a physical quantity with its corresponding unit, and is akin to asking, “What is the kilogrammage of that boulder?” rather than “What is the mass of that boulder?” For an aspiring STEM professional, use of the word “amperage” is a bad habit which should be avoided.)

It is traditional to use the variable *q* to represent charge, and *i* to represent current. (Beware that electrical engineers use *j* to represent , so as not to confuse this with current *i*.) The above equation, written in terms of these variables, becomes



Electric charges move around a circuit due to differences in electrical potential energy, just as water in a river moves around due to differences in gravitational potential energy. Thinking in terms of gravitational potential energy, suppose you were attending a conference in a hilly part of San Francisco. Early one morning you decided to exercise by walking around the block. Your gravitational potential energy increased and decreased as you walked up and down the hills. When you returned to your hotel, you were at the same altitude or elevation as when you left. So, despite all the ups and downs involved in the walk, there was *zero* net change in your potential energy after you added all the positive (uphill) and negative (downhill) changes along the way, provided that you traversed a complete loop.

Electric charges in a closed-loop circuit are subject to a similar rule. Passing through a circuit element (resistor, capacitor, inductor, or power supply) increases or decreases their electrical potential energy, according to simple rules that are well understood for each type of circuit element. You may think of these energy changes as “uphill” (positive) or “downhill” (negative). However, an electron that returns to its starting point in a loop circuit has the same potential energy at the end of its “walk around the block” as it did when it started; it’s at the same location, so it has the same “electrical elevation.” Our model will therefore require the sum of all potential energy changes around a closed-loop circuit to be zero (Kirchhoff’s loop theorem). In the language normally used to describe electrical circuits, these potential energy changes (even the “uphill” or positive ones) are called ***voltage drops***.

The loop in Figure 1.3.4(a) of the textbook [Zill] is a pictorial representation of an *LRC* series circuit. These letters come from the names of the circuit elements arranged around the loop; *L* = ***inductor***, *R* = ***resistor***, and *C* = ***capacitor***. An additional element (the power supply, labeled “”) is always assumed to be present in a circuit of any type, so its label “*E*” does not appear in the name of this circuit.

Each circuit element is also assigned a positive numeric value that describes its electromagnetic behavior in the circuit; going back to the San Francisco analogy, each circuit element corresponds to a hill (up or down), and you may think of these numbers as describing the steepness (although not the height) of each circuit element’s “hill.” The inductor is a coil of wire (hence the shape of its symbol in the figure) that magnetizes when current passes through it; its ***inductance*** *L* is a constant indicating how strong the magnetic field is for a given level of current. The resistor’s purpose is to restrict the amount of current in the circuit (the resistor may get hot so the rest of the circuit doesn’t); its ability to limit the current is described by the constant value *R* of its ***resistance***. The capacitor provides temporary charge storage (a Leyden jar is one type of capacitor). Its ***capacitance*** *C* is a constant that describes how many coulombs of charge it could store if it were connected to a 1-volt battery. The power supply or battery is characterized by its ***electromotive force*** *E*, measured in ***volts***. The electromotive force may be a constant (DC) or variable (AC) function of time.

From a physical analysis of the properties of each circuit element (due in large part to the work of Michael Faraday in the nineteenth century), it is possible to obtain formulas for the voltage drops of each element. Again, returning to the San Francisco analogy, the voltage drops correspond to the heights of each energy “hill” in the circuit. These formulas turn out to be:

Inductor: 

Resistor: 

Capacitor: 

Power supply: 

In the first two formulas above, *i* is changed to , as we defined earlier, so that there is only one dependent variable in the model rather than two. Because the sum of these voltage drops must be zero, our model suggests that

 or 

One last note on circuits (for the related homework problems): If the inductor is removed from the circuit, the value of *L* goes to approximately zero (assuming the loose ends of the wires are reconnected). If the resistor is removed, the value of *R* is also approximately zero. However, if the capacitor is removed, the value of *C* approaches infinity. This oddity results because the capacitance *C* is proportional to the amount of charge a DC power supply can pump into the circuit before it fills the storage capacity of the capacitor. With the capacitor gone, there is no restriction as to how much charge the power supply can pump out (the charge just circulates around the circuit indefinitely through the remaining inductor and/or resistor), so the effective capacitance goes to infinity.

Please read about the other models describe in this section of the textbook to give you an idea of the broad range of systems and phenomena to which differential equations may be applied, and note that they focus on developing the corresponding differential equations, but not (yet) solving them. We will re-encounter some of these situations later in the course, after we learn to solve the corresponding differential equations, and use the solutions to help us characterize the behavior of the systems.