MATH V23 LECTURE NOTES (Bowen)
Section 1.2
Initial Value Problems

Unless otherwise stated, it will be assumed in this section that *x* is the independent variable, and *y* is the dependent variable, for each differential equation discussed.

Initial conditions and initial value problems. Suppose we are given an ODE of order *n*, together with a specification of the numerical value  of the solution corresponding to a given value  of the independent variable (that is, ). Further suppose that we are given additional specifications , , , … ,  at the same point, or a total of *n* specifications in total. Taken together, the specifications of the numeric values of *y* and its first  derivatives evaluated at  are called ***initial conditions***, and the entire problem is called an ***initial value problem*** (IVP). The terminology arises from physical problems modeled by first- or second-order ODEs in which these designations provide the initial position (or position and velocity) of a particle.

Interval of validity for IVPs. The interval of validity for an IVP may differ from the interval of validity for the general solution of an ODE. (The ***general solution*** does not take the initial conditions into account, but the ***actual solution*** does.) Consider, for example, the IVP

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Show that the general solution (a one-parameter family) is . The actual solution is found by determining the value of the parameter *c* that satisfies the initial condition :



This gives an actual solution of  for the IVP.

If we think of the *function*, its domain includes all real *x*, excepting . If we think of the largest possible *interval of validity for the general solution* , it is . But the *interval of validity for the IVP* is just , because the given value of (the 6 from the initial condition ) lies on this subinterval, but not on the subinterval . This illustrates that the interval of validity of an IVP is not necessarily the same as the interval of validity for the corresponding general solution, as the interval for the IVP must contain the given value  of the independent variable taken from the statement of initial conditions. Note (*e.g.*, Example 4 in the text) that an IVP may still have multiple solutions.

Unique solution of a first-order IVP. A question of great importance in the advanced study of differential equations is determining how many solutions the equation has. The answer can range from “zero” (*i.e.*, don’t bother trying to solve this DE; it has already been proved that no solution exists) to “infinitely many” (*i.e.*, are you sure you’ve found *all* the solutions?), depending upon the type of equation. Several theorems have been proved about the number of solutions associated with certain specific types of DEs.

The first such theorem in this course is Theorem 1.2.1; it has a somewhat lengthy statement, but it boils down to this: If you have a first-order IVP of the form  subject to , and  and  are continuous at and near , then it has exactly *one* actual solution whose interval of validity contains .

In some ways, this is a nice theorem, because if you can find just one actual solution to a first-order IVP that meets the reasonable conditions stated, you may be assured that you have not accidentally overlooked the existence of other possible solutions, provided you stay near . However, it also has a limitation in that, as you move away from  (perhaps even just a tiny distance such as 0.000001), it becomes possible for more than one solution of the IVP to exist. So, you should use this theorem with caution, and not assume that your solution is unique everywhere on the *x*-axis just because it satisfies the hypothesis of this theorem. Another way to look at this is to realize that the interval on which a solution is *unique* may differ from the interval on which at least one solution *exists*.

There is a more complicated uniqueness theorem for second-order equations that we will address later in the course.