MATH V23 LECTURE NOTES (Bowen)
Section 1.1
Definitions and Terminology

Differential equations. A ***differential equation*** (DE) is an equation containing first, second, or higher derivatives of one or more unknown functions of one or more independent variables. In most cases, we seek the solution(s) of the equation. A ***solution*** is a specific function which, when substituted into the original DE, satisfies that equation (*i.e.*, makes it true). Often, a solution is only valid for certain values of the independent variable(s); if so, these values (*i.e.*, the interval of validity of the solution) must be stated in addition to the functional form(s) of the solutions. If a solution’s interval of validity were , then the corresponding solution would be valid only for .

Solutions. It is often impossible to find closed-form solutions to DEs. (A ***closed-form solution*** is an expression written in terms of familiar algebraic functions, such as polynomials, rational functions, exponential/logarithmic functions, or trigonometric/inverse trigonometric functions). In some cases, a solution still exists, but can only be expressed as a series (*e.g.*, as a power series) or in graphical form. For example, the DE



can only be solved using a graph or power series; there is no closed-form solution, because  does not possess a generic closed-form antiderivative. Both graphing and series solutions will be addressed in subsequent sections.

Notation. The author of our textbook uses (more or less interchangeably) the notations , , or *y* to represent functions or dependent variables. (The most common independent variables are *x* or *t*.) For those of you who have not yet completed MATH V21C,  is the Greek letter “phi”, which is correctly pronounced “fee,” but is often pronounced so it rhymes with . (If you see an English word in which “ph” is used in place of “f,” such as “physics,” “photon,” “phobia,” or “elephant,” it was probably derived from a Greek word containing .) Do not confuse  with “theta” . In calculus and physics, , like , usually represents an angle, but in this course,  will generally represent a function, and you should *not* assume it to be an angular quantity. Derivatives may be represented using Leibniz notation , Lagrange notation , or Euler notation ; the latter may be new to you. When the independent variable is time, Newton’s dot notation may also be used . The second derivative using each of these notations may be written , , , and , respectively. For fourth derivatives or higher, the Lagrange notation is usually written as  rather than as .

Type of a DE. Consider ordinary differential equations (ODE) (one independent variable) versus partial differential equations (PDE) (two or more independent variables). This course focuses on ODEs; however, you might see PDEs in your physics course on waves. For example, the general three-dimensional wave equation is a PDE:



where *f* is the amplitude of the wave at a given place  and time *t*, and *c* (taken from the first letter of the Latin *celeritas*, meaning “speed”) indicates the magnitude of the wave’s velocity. For students who have not taken MATH V21C, the curly  is a ***partial derivative***, used in multivariable calculus to indicate a derivative taken with respect to one of several independent variables while all the remaining independent variables are held constant.

Order of a DE. The ***order*** of an ODE or PDE is the order of the highest derivative in the equation. For example, if an equation contains the first and third derivatives of the dependent variable, then it is a third-order DE. The wave equation above is a second-order PDE. Higher-order equations are generally more difficult to solve than lower-order equations.

Forms of an ODE. If an ODE is first-order, such as



then it may be rewritten using the Leibniz notation as



and then multiplied everywhere by *dx* to yield the so-called ***differential form***, which in this case would be



In general, the differential form is written as



For the example above,  and , and the preceding form is often abbreviated as .

Two other forms in which an ODE of any order *n* may be written are the ***general form***



and the ***normal form***



which may be derived from the general form by algebraically isolating the highest derivative in the equation.

Linearity and homogeneity. An ODE is called ***linear*** if it can be written in the form



where each  is either a function of *x* only, or a constant. Otherwise it is called ***nonlinear***. (The textbook uses  in its definition, but this may be confusing because most people think of  as a constant coefficient. However, the  notation makes sense if you think of the  as a finite *sequence* of functions, where “sequence” is used with the meaning given in section 11.1 of the calculus textbook [Stewart].) In general, nonlinear equations are much more difficult to solve than linear equations. This course will focus mostly on linear ODEs.

If it turns out that  above, then the linear equation is also said to be ***homogeneous***. We will later discover that there is a relationship between homogeneous linear ODEs and homogeneous matrix equations in linear algebra. So, the use of this word in both contexts to describe an equation that “ends in zero” is not just a coincidence. One parallel is that a homogeneous linear ODE always has at least one solution (the trivial solution ), just as a homogeneous matrix equation  always has at least one solution (the trivial solution ).

Definition of solution. Consider an ODE of order *n*. If, on some real interval *I*, there is a function  which is continuous on *I*, whose first *n* derivatives are also continuous on *I*, and which, when substituted into the ODE, reduce it to an identity for all , then  is a ***solution*** of the ODE.

The *solution*  may differ from the *function*  because of discrepancies in their domains. For example, the domain of the absolute value function  (as we usually think of it) is . If we identify this function as a solution of the ODE , however, it only “works” on the interval , because the absolute value function has the wrong slope  for ; moreover, the slope is undefined at  due to the sharp bend at that point in the graph of . For this reason, it is usually insufficient to specify just a function’s equation when stating the solution to an ODE; an interval of validity must also be provided, unless the solution works for all real *x*. The stated interval of validity should be the largest possible interval (or union of intervals) *I* on which the solution works.

Explicit or implicit solutions. A solution may be explicit or implicit. An ***explicit*** solution generally is written with the dependent variable isolated on one side of the equation; for example, . An implicit solution has the independent and dependent variables “mixed” throughout the equation; for example, . As a rule of thumb, an implicit solution would be an expression on which one would use the technique of implicit differentiation if it became necessary to differentiate the expression (review Stewart section 3.5 if necessary).

Families of solutions, particular solutions, and singular solutions. For the same reason that a function has an infinite number of antiderivatives (the constant *C* added to the end whenever we integrate a function), an ODE typically has a multitude of solutions. A set of solutions that differ from each other only in the values of one or more constants (called ***parameters***) is called a ***family of solutions***. The members of a family of solutions tend to have graphs that are similar in shape, but differ by a shift, stretch, and/or reflection. Selecting specific numerical values for each of the parameters has the effect of singling out an individual family member; this member is called a ***particular solution***. If an ODE is specified with one or more ***initial conditions***, such as  or , the additional information may (or may not) be sufficient to select a particular solution. Without information regarding initial conditions, we may only be able to identify families of solutions.

An ODE may have additional solutions that are not part of any family of solutions (that is, they cannot be generated by substituting any of the possible parameter values into the general equation describing a family of solutions). These additional solutions are called ***singular solutions***.

For example, suppose that *x* is the independent variable, and *y* the dependent variable. Consider the nonlinear ODE



It turns out that a one-parameter family of solutions, valid for all real *x* () has the general form , where *c* (the parameter) may be set equal to any real number. We may show that each instance of  is a solution of the ODE. Find its derivative (using the chain rule)



and plug both  and  into the original equation to show that they reduce it to an identity:



Another solution, , can be verified to work (again, by direct substitution), but it cannot be obtained using any value of *c* in the family of solutions  illustrated above. Therefore  is a singular solution.

Piecewise-defined solutions. Show (using direct substitution) that a one-parameter family of nontrivial solutions for the second-order linear homogeneous ODE



which is valid for all real *x*, is given by



However, these are not the only solutions. Consider the piecewise function



whose first and second derivatives are

 and 

Note that the two “pieces”  and  of , taken individually, are each valid solutions on  (they correspond to the solution family members  and , respectively), so  is a valid solution on the subinterval , and  is a valid solution on the subinterval . Moreover, the functions , , and  all pass through the origin, so they are defined and continuous at the “join” point . Therefore, the piecewise function  is also a solution of the original equation for all real *x*, even though it cannot be obtained from any single member of the family of solutions .

Solutions that are non-elementary integrals. In some instances, the solution of an ODE can be a non-elementary integral (that is, the value of a definite integral of a function whose antiderivative cannot be expressed in closed form). Consider, for example, the linear ODE



We claim that a one-parameter family of solutions is



where the parameter *c* appears in the lower limit of integration. The integral is impossible to evaluate in closed form, because the function  (which should not be confused with ) does not have an antiderivative that can be written in terms of elementary functions. In such cases, we may use the Fundamental Theorem of Calculus to verify the correctness of this result. One statement of the Fundamental Theorem (see section 5.3 of Stewart) is that for a function  that is continuous on , the derivative of



is guaranteed to be  for all *x* on . In the above example,  is  (so  is ), and  is . It immediately follows from the Fundamental Theorem that , thus completing the verification of the claim that  is a solution of the initial ODE.