FACTORING AND THE BINOMIAL THEOREM

Factoring converts expressions written as **terms** into equivalent expressions written as **factors**.

GCF method[*]: $3x^5 - 8x^6 = \underline{\underline{x^5(3-8x)}}$; but compare/contrast with $3x^{-5} - 8x^{-6} = \underline{\underline{x^{-6}(3x-8)}}$.[†]

Grouping method: $3x^2 - 5x - 12x + 20 = x\underline{(3x-5)} - 4\underline{(3x-5)} = \underline{\underline{(3x-5)(x-4)}}$

"ac" method: $\begin{cases} 3x^2 - 17x + 20: \;\; ac = (3)(20) = 60; \;\; \text{factors of 60 are } (\pm1)\cdot(\pm60), (\pm2)\cdot(\pm30), (\pm3)\cdot(\pm20), \\ (\pm4)\cdot(\pm15), (\pm5)\cdot(\pm12), (\pm6)\cdot(\pm10); \;\; \text{the pair } (-5) \;\&\; (-12) \text{ add to } b = -17, \\ \text{so break up } 3x^2 - 17x + 20 \text{ into } 3x^2 - 5x - 12x + 20; \;\; \text{finish by using the grouping method.} \end{cases}$

Special cases:

Difference of squares: $a^2 - b^2 = \underline{\underline{(a+b)(a-b)}}$

Sum of squares: $a^2 + b^2 = \underline{\underline{\text{prime}}}$ (cannot be factored without i)[‡]

Sum/diff. of cubes: $a^3 \pm b^3 = \underline{\underline{(a \pm b)(a^2 \mp ab + b^2)}}$ ($\mp$ means ***reverse sign*** between a & b)

The **binomial theorem** allows large powers of binomials to be determined without tedious FOILing.

General expansion for whole-number exponents *n*:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ and $n! = (n)\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(3)\cdot(2)\cdot(1)$. Note: $n! = n\cdot(n-1)!$ for $n \geq 1$, so $\underline{\underline{0! = 1}}$.

Whole-number *n*: Let $a = 3x$, $b = -2y$, and $n = 4$. Then $(3x-2y)^4 = [3x + (-2y)]^4 = \sum_{k=0}^{4} \binom{4}{k} (3x)^{4-k} (-2y)^k$

$$(3x-2y)^4 = \binom{4}{0}(3x)^{4-0}(-2y)^0 + \binom{4}{1}(3x)^{4-1}(-2y)^1 + \binom{4}{2}(3x)^{4-2}(-2y)^2 + \binom{4}{3}(3x)^{4-3}(-2y)^3 + \binom{4}{4}(3x)^{4-4}(-2y)^4$$

$$(3x-2y)^4 = \frac{4!}{0!4!}(3x)^4 + \frac{4!}{1!3!}(3x)^3(-2y) + \frac{4!}{2!2!}(3x)^2(-2y)^2 + \frac{4!}{3!1!}(3x)^1(-2y)^3 + \frac{4!}{4!0!}(-2y)^4$$

$$(3x-2y)^4 = \frac{4!}{4!}(3)^4(x)^4 + \frac{4\cdot3!}{1\cdot3!}(3)^3(x)^3(-2y) + \frac{4\cdot3\cdot2!}{2\cdot2!}(3)^2(x)^2(-2)^2(y)^2 + \frac{4\cdot3!}{3!\cdot1}(3)^1(x)^1(-2)^3(y)^3 + (-2)^4(y)^4$$

$$(3x-2y)^4 = \underline{\underline{81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4}}$$

Handy identities: $\binom{n}{0} = 1$; $\binom{n}{1} = n$; $\binom{n}{2} = \dfrac{n(n-1)}{2}$; $\binom{n}{3} = \dfrac{n(n-1)(n-2)}{3!}$ (valid for all $n \neq 0$)

Extension formula for negative or rational *n*: For negative or rational n, the factorials cannot be computed. However, the **handy identities** above allow us to approximate the sum with the first few terms if $b \ll a$:

$$(a+b)^n \approx a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{6}a^{n-3}b^3 + \ldots (\textit{terms never end})$$

Example, rational *n*:

$$1.05^{-3/8} = (1+0.05)^{-3/8} \approx (1)^{-3/8} + \left(-\tfrac{3}{8}\right)(1)^{-11/8}(0.05) + \frac{\left(-\frac{3}{8}\right)\left(-\frac{11}{8}\right)}{2}(1)^{-19/8}(0.05)^2 + \ldots$$

$$1.05^{-3/8} \approx (1) - \left(\tfrac{3}{8}\right)(0.05) + \frac{\left(\frac{33}{64}\right)}{2}(0.05)^2 = 1 - \left(\tfrac{3}{8}\right)(0.05) + \left(\tfrac{33}{128}\right)(0.0025) \approx \underline{\underline{0.981894}}.$$[§]

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[*] ***Always*** try the GCF method ***before*** trying any of the other methods.

[†] It is always the ***smallest*** power of each variable or parentheses that is factored out, regardless of whether it is positive or negative.

[‡] If we allow for complex numbers, then $a^2 + b^2 = (a+ib)(a-ib)$. However, in this course we will only deal with real numbers.

[§] By comparison, a calculator gives $1.05^{-3/8} \approx 0.981870$. Above, we set $a = 1$, $b = 0.05$, and $n = -\frac{3}{8}$, and used the **extension formula**.

Problems: Work on another sheet of paper and ***turn in*** with your other homework before the first exam. Show all steps. Solve problems 1–5 using factoring methods only, and problems 6–10 using the binomial theorem or approximations.

1. Factor as completely as possible: $9x^2 - 36y^2$.

2. Factor as completely as possible: $4x^2 - 22x - 42$. Do not solve for *x*; just show your parentheses.

3. Factor as completely as possible: $x^3 - 5x^2 - 6x + 30$. Do not solve for *x*; just show your parentheses.

4. Factor as completely as possible: $x(2x-5)^{-3/2} - 4x^2(2x-5)^{-5/2}$. Note that the parentheses are very similar.

5. Use the difference-of-squares formula in reverse to compute the product 87×93 ***without*** using a calculator or the usual multiplication algorithm. Hint: $(90-3)(90+3)$ has the same structure as $(a-b)(a+b)$. After you finish, compute 87×93 ***with*** a calculator and note whether the results are comparable.

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6. Use the binomial theorem to find the ***complete*** expansion of $(x+y)^6$.

7. Use the binomial theorem to find the ***complete*** expansion of $(4x-7y)^5$. (See **Whole-number *n*** on the front of this sheet for a similar example.) Be cognizant of how the minus sign will affect the result (treat it as a -1 and raise it to even or odd powers). Do not be intimidated by the five-digit coefficients you may obtain (use your calculator).

8. Estimate the value of 5.1^4 without using a calculator. Write only the ***first three terms*** of the binomial expansion of $(a+b)^4$, then substitute $a = 5$, $b = 0.1$, and $n = 4$, and multiply or add the resulting numbers by hand. Compare with the actual result obtained from a calculator and note whether the results are comparable.

9. Estimate the value of $1.02^{-1/2}$ without using a calculator. Write only the ***first three terms*** of the binomial expansion of $(a+b)^{-1/2}$ (use the **handy identities** on the front of this sheet in place of the binomial coefficients, since you can't compute the factorials here). Then substitute $a = 1$, $b = 0.02$, and $n = -\frac{1}{2}$, and multiply or add the resulting numbers by hand. Hint: $0.02^2 \neq 0.04$; think of this as $\frac{2}{100} \cdot \frac{2}{100}$ and then convert the fractional product back to decimal. Compare with the actual result obtained from a calculator; are they close?

10. (Optional extra credit) In the study of special relativity, velocities are frequently designated using the Greek letter β (beta). If a 24th century starship were to travel at 95% of the speed of light, then $\beta = 0.95$; at 99.9% of the speed of light, $\beta = 0.999$. (β has no units; if you want the speed *v* in km/s, use $v = \beta c$, where *c* is the speed of light.) As objects approach *c*, relativity predicts that their masses increase; another Greek letter γ (gamma) is used to represent the mass increase, relative to the original mass at rest (if $\gamma = 2$, the object's speed is so great that its mass has doubled, compared to what it was at rest). The exact formula to obtain γ at any speed turns out to be $\underline{\underline{\gamma = (1-\beta^2)^{-1/2}}}$. Typical NASA interplanetary probes of the early 21st century traveling at 30 km/s have a very small $\beta \approx 0.0001$, so their relativistic mass increase is quite small. (At 30 km/s, you could fly from LA to New York in well under three minutes, yet this speed is still tiny compared to *c*.)

(a) Verify, by direct substitution, that for an object at rest ($\beta = 0$), γ is exactly equal to 1.

(b) Find, by direct substitution, what value of β would have to be achieved to double the mass of an object (you should obtain a square root). Calculate the decimal value of this square root; verify that $\beta < 1$.

(c) Use the ***first two terms*** of the binomial expansion (employ the **handy identities**, not factorials) to show that for a "slow" NASA interplanetary probe, a good approximation for γ is $\gamma \approx 1 + \frac{1}{2}\beta^2$. (Since $\beta \ll 1$, expand $(a+b)^n$ using $a = 1$, $b = -\beta^2$, and $n = -\frac{1}{2}$, but ***stop*** after the first two terms.)

(d) Use the result of part **(c)** to estimate the ***numerical*** value of γ for a typical early 21st century interplanetary probe. Do ***not*** round the result (use all the significant figures available on your calculator display). Provide a brief interpretation of your answer (what does it say about the mass of the NASA probe while it is in motion?).